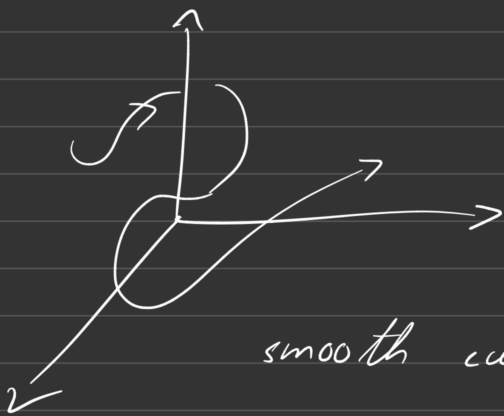
$\mathbb{R}^3$



$I = (a, b)$ interval

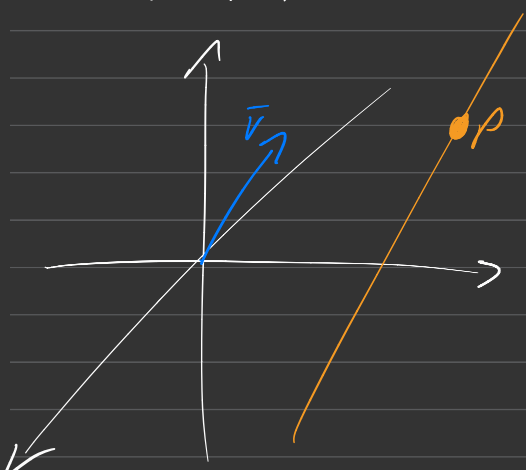
$\alpha : I \longrightarrow \mathbb{R}^3$
$t \longmapsto (\alpha_1(t), \alpha_2(t), \alpha_3(t))$

smooth curve means

$\alpha_1, \alpha_2, \alpha_3 : I \to \mathbb{R}$ smooth (differentiable)

~~Example~~

$p \in \mathbb{R}^3$, $\bar{v} \in \mathbb{R}^3$
$\shortparallel$
(p_1, p_2, p_3)



$\alpha(t) = p + t\bar{v}$

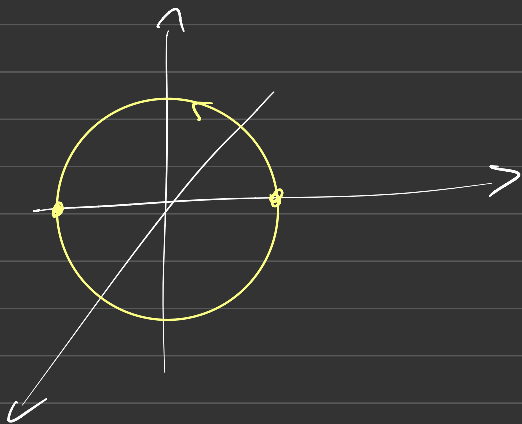
$(t \in (a, b) \subset \mathbb{R}$

$$\alpha(t) = \begin{pmatrix} p_1 + tv_1 \\ p_2 + tv_2 \\ p_3 + tv_3 \end{pmatrix}$$

planar

Examples

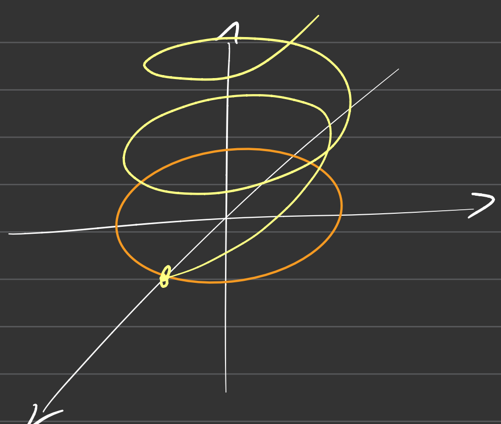
$\alpha(t) = (0, \cos t, \sin t)$



planar

Examples

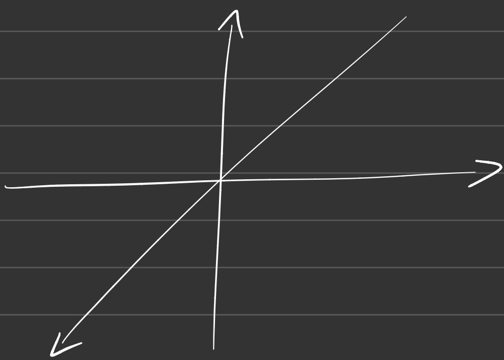
$\alpha(t) = (\cos t, \sin t,$



non-planar

Example

$\alpha(t) = (t, t^2, t^3)$



funny shaped line

non-planar

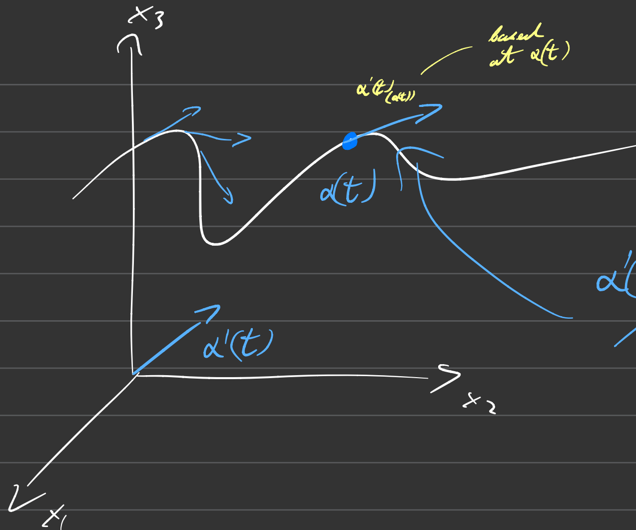
Working with some arbitrary (smooth) curve $\alpha : I \to \mathbb{R}^3$, $t \mapsto (\alpha_1(t), \alpha_2(t), \alpha_3(t))$

Definition

For each $t \in I$, we have a vector

$$\alpha'(t) := (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t))$$

called the velocity vector of α at t



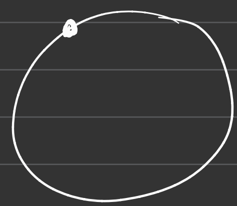
$\alpha'(t)$ tangent to α at t

$|\alpha'(t)|$ = speed of α at t

The curve α is said to be <u>regular</u> if

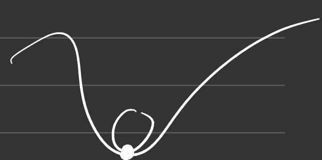
$$|\alpha'(t)| \neq 0 \quad \forall\ t \in I$$

Henceforth all curves are assumed to be smooth and regular



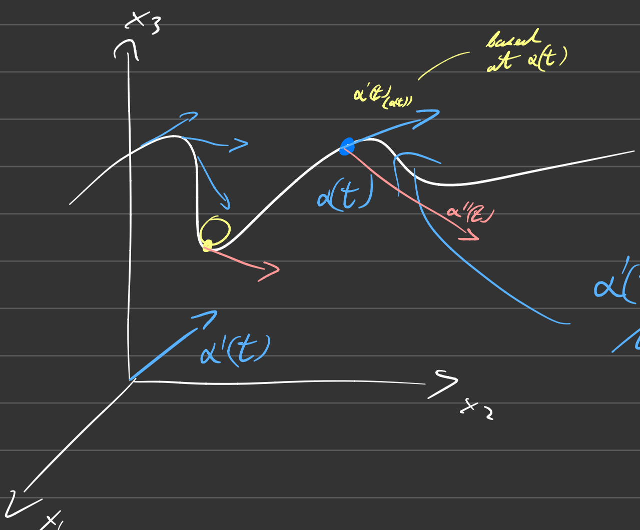
$\kappa = \frac{1}{r}$

curv



Definition

For each $t \in I$, we have a vector
$$\alpha''(t) := (\alpha_1''(t), \alpha_2''(t), \alpha_3''(t))$$
called the **acceleration vector** of α at t



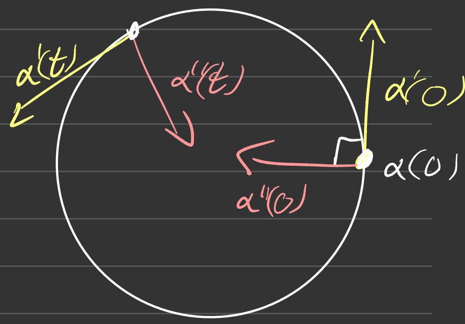
$\alpha'(t)$ tangent to α at t

Example

$$\alpha(t) = (\cos t, \sin t)$$

$$\alpha'(t) = (-\sin t, \cos t)$$

$$\alpha''(t) = (-\cos t, -\sin t)$$



$$\alpha'(t) \cdot \alpha''(t) = 0$$

Example

$\alpha(t) = (t, t^2, t^3)$

$\alpha'(t) = (1, 2t, 3t^2)$

$\alpha''(t) = (0, 2, 6t)$

$\alpha'(t) \cdot \alpha''(t) = 0 + 4t + 18t^3$

Length $|\alpha'(t)| = \sqrt{1 + 4t^2 + 9t^4}$

non-constant

We say $\alpha(t)$ is a constant speed curve if

$|\alpha'(t)| = c$ constant $\forall\ t \in I$,

A unit speed curve if

$|\alpha'(t)| = 1 \quad \forall\ t$

Lemma

Suppose α is a constant speed curve then

$$\alpha'(t) \perp \alpha''(t)$$

Proof

Suppose $|\alpha'(t)| = c > 0$ constant

$$\alpha'(t) \cdot \alpha'(t) = c^2 \quad \text{const}$$

$$\frac{d}{dt}(\alpha'(t) \cdot \alpha'(t)) = \frac{d}{dt}(c^2) = 0$$

Leibnitz

$$(\alpha'(t) \cdot \alpha''(t) + \alpha''(t) \cdot \alpha'(t))$$

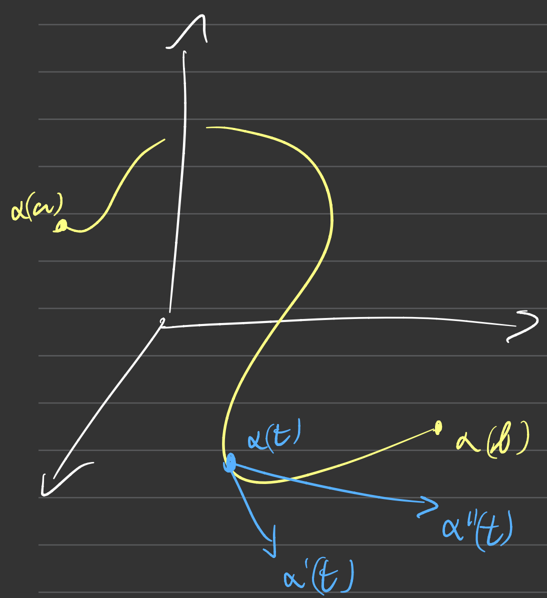
$$= 2\,\alpha'(t) \cdot \alpha''(t) = 0$$

$I = [a, b]$

$\alpha : I \longrightarrow \mathbb{R}^3$ smooth curve
$t \longmapsto (\alpha_1(t), \alpha_2(t), \alpha_3(t))$

Velocity vector $\alpha'(t) = (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t))$

Acceleration $\alpha''(t) = (\alpha_1''(t), \alpha_2''(t), \alpha_3''(t))$



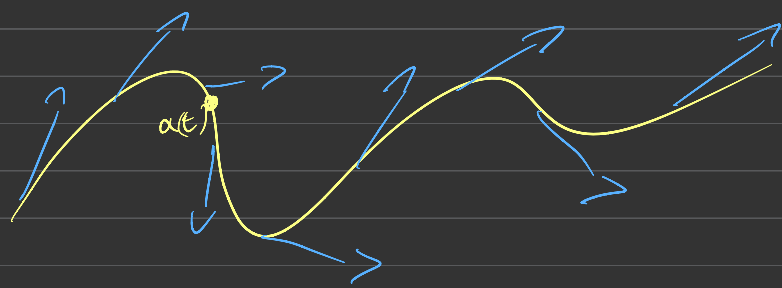
A vector field X along α is a smooth map

$X : I \longrightarrow \mathbb{R}^3$

which associats for each $t \in I$ a vector

$X(t) = (X_1(t), X_2(t), X_3(t))$

"based" at the point $\alpha(t)$



Leibnitz Rule

$X, Y : I \longrightarrow \mathbb{R}^3$ are vector fields along α

$$X \cdot Y : I \longrightarrow \mathbb{R}$$
$$t \longmapsto X(t) \cdot Y(t)$$

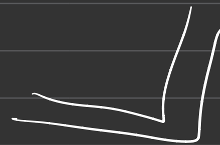
dot product

$$(X \cdot Y)' = \frac{d}{dx}(X \cdot Y) = X' \cdot Y + X \cdot Y'$$

$$x'(t) = (x_1' t, x_2'(t), x_3'(t))$$

Terminology

X is parallel along α if
$X' \equiv (0,0,0)$



Recall

Regularity condition

The curve $\alpha : I \to \mathbb{R}^3$ is regular if $\underbrace{|\alpha'(t)|}_{\text{speed}} > 0$ for all t

Lemma

Suppose α is a regular constant speed curve, so

$|\alpha'(t)| = c > 0$, (constant)

Then $\alpha'(t) \cdot \alpha''(t) = 0 \quad \forall t$

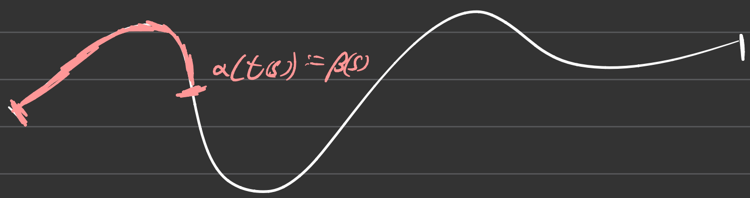
$\alpha' \perp \alpha''$

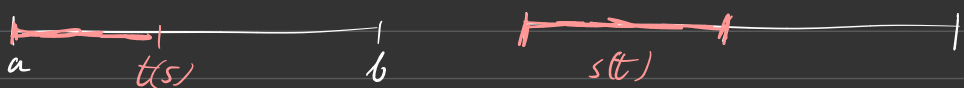
In the case when $|\alpha'(t)| \equiv 1$ $(\forall t)$ we say α is unit speed curve (u.s.c)

Storytime

$\alpha : I \longrightarrow \mathbb{R}^3$ smooth reg curve

$$|\alpha'(t)| > 0 \quad \forall t$$





Let $s(t)$ denote length of α over $[0,t]$

$s : [a,b] \longrightarrow [0,L]$

where L = length of curve $\alpha = s(b)$

$$s(t) = \int_a^t |\alpha'(u)|\,du$$

$$s'(t) = \frac{d}{dt}\left(\int_a^t |\alpha'(u)|\,du\right) = |\alpha'(t)| > 0$$

Thus s is increasing and so injective (as well as surjective)

Denote inverse by $t(s)$

Define $\beta(s) := \alpha(t(s))$

$\beta : [0, L] \to \mathbb{R}^3$

$$|\beta'(s)| = |\alpha'(t(s))\, t'(s)|$$

$$= \frac{|\alpha'(t(s))|}{|s'(t)|}$$

$$= \frac{|\alpha'(t)|}{|\alpha'(t)|} = 1$$

Lemma

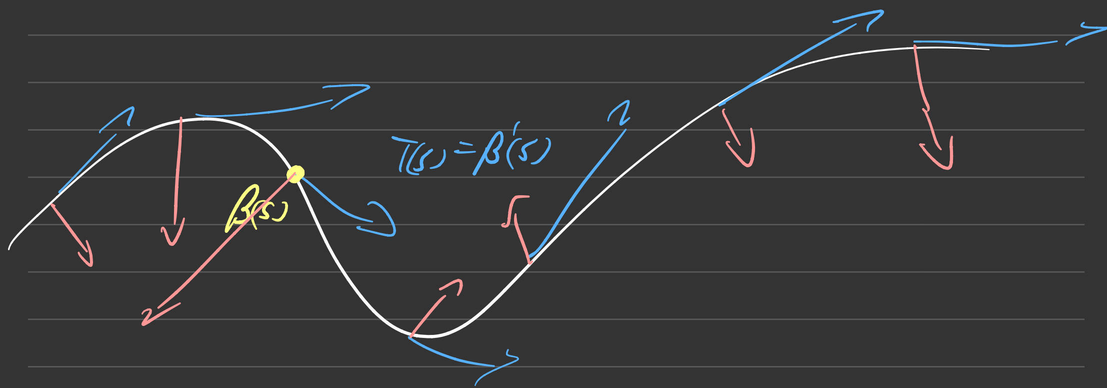
Every smooth regular curve $\alpha : I \to \mathbb{R}^3$ has a unit speed reparameterisation

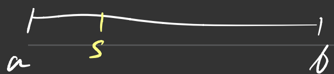
(arc-length parameterisation)

Henceforth assume $\beta : I \to \mathbb{R}^3$ is a unit speed curve

Standard notation: $\boxed{T(s) = \beta'(s)}$, the unit tangent vec field

$\boxed{\text{Define } \kappa(s) := |T'(s)| = |\beta''(s)|,}$ the curvature of β at s





Example

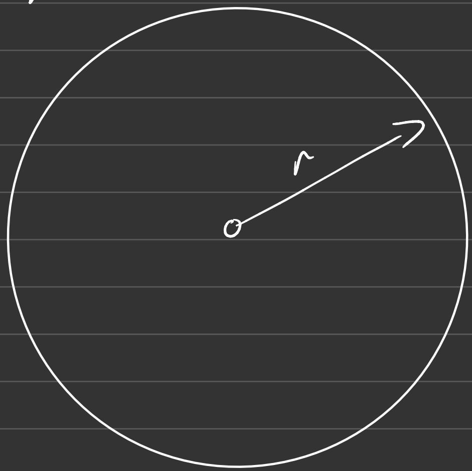
$$\beta(s) = \left(r\cos\frac{s}{r},\ r\sin\frac{s}{r},\ 0\right)$$

$$\beta'(s) = \left(-\frac{1}{r} r\sin\left(\frac{s}{r}\right),\ \frac{1}{r} r\cos\left(\frac{s}{r}\right),\ 0\right)$$

$$= \left(\sin\left(\frac{s}{r}\right),\ \cos\frac{s}{r},\ 0\right) \qquad \text{(unit length)}$$

$\beta''(s) = \left(-\frac{1}{r}\cos\left(\frac{s}{r}\right), -\frac{1}{r}\sin\left(\frac{s}{r}\right), 0\right)$

$\kappa(s) = \frac{1}{r}$



Proposition

α is a straight line $\Leftrightarrow \kappa \equiv 0$

$\kappa \equiv 0 \Leftrightarrow |\beta''(s)| = 0$

$\Leftrightarrow \beta''(s) = (0,0,0)$

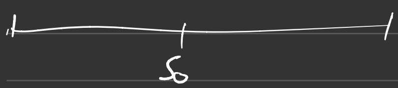
$\Leftrightarrow \beta'(s) = (v_1, v_2, v_3) =$ constant vector

$\Leftrightarrow \beta(s) = p + t\bar{v}$ for some $p \in \mathbb{R}^3$

We will usually consider the following situation

$\beta : I \longrightarrow \mathbb{R}^3$ smooth unit speed curve with

$k(s) > 0 \quad \forall\, s \in I$



- k is the rate at which T is turning, thats why T has to be the same size and only direction can change.

- We are interested in β unit speed such that

$$k(s) > 0 \quad \forall \quad s \in I$$

In this case $\quad T'(s) \neq (0,0,0) \quad \forall s$

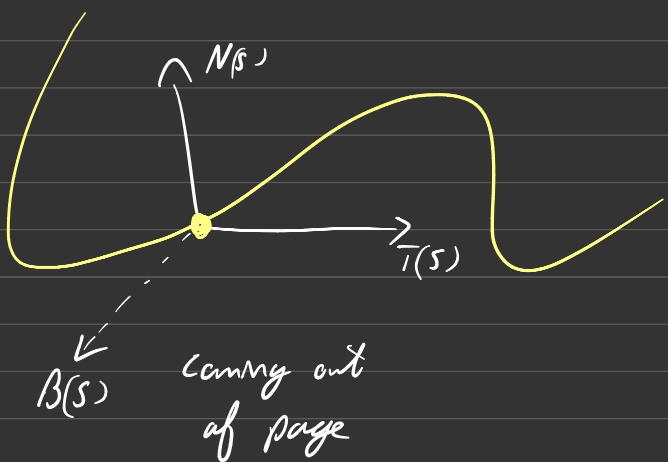
We know $\quad T'(s) \perp T(s) \quad \forall s$

Define $N(s) = \dfrac{T'(s)}{|T'(s)|} = \dfrac{T'(s)}{\kappa(s)}$

N is a unit vector field orthogonal to T called the principal normal vector field.

Finally define

$B(s) := T(s) \times N(s)$ binomial vector field



The triple vector fields (T, N, B), provides a smoothly varying orthonormal basis

basis $(T(s), N(s), B(s))$

for $\mathbb{R}^3$ at each s. Such a basis is called a frame and the triple (T, N, B) is called the Freset frame field

Theorem (Frenet-Serret Formulae)

$\beta : I \longrightarrow \mathbb{R}^3$, unit speed curve

$\kappa(s) > 0 \quad \forall s$

$N(s) = \frac{T'(s)}{|T'(s)|} = \frac{T'(s)}{\kappa(s)}$

$\tau(s) = -B'(s) \cdot N(s)$

$B(s) = T(s) \times N(s)$

(i) $T'(s) = \kappa(s) N(s)$

(ii) $N'(s) = -\kappa(s) T(s) + \tau(s) B(s)$

(iii) $B'(s) = -\tau(s) N(s)$

Where $\tau : I \longrightarrow \mathbb{R}^3$ is a smooth function called the torsion of β ($\tau(s)$ = torsion of β at s)

~~Proof~~

(i) True by definition

(ii) T, N, B is orthonormal basis $\forall s$ we can write

$$B'(s) = (B'(s) \cdot T(s)) T(s) + (B'(s) \cdot N(s)) N(s)$$
$$+ (B'(s) \cdot B(s)) B(s)$$

But $B(s) \cdot T(s) = 0$

$$B'(s)\cdot T(s) + B(s)\cdot T'(s) = 0$$

$$\Rightarrow \quad B'(s)\cdot T(s) = -B(s)\,T'(s)$$

$$= -B(s)\cdot(\kappa(s)\,N(s))$$

but since $B \perp N = 0$

$$B(s)\cdot B(s) = 1, \quad 2B(s)\cdot B'(s) = 0$$

Hence $B'(s) = (B'(s)\cdot N(s))\,N(s)$

so we define

$$\tau(s) = -B'(s)\cdot N(s)$$

(iii) $N\cdot N = 1 \quad \Rightarrow \quad N\cdot N' = 0$

$$N\cdot T = 0$$

$$N'\cdot T + N\cdot T' = 0 \quad \Rightarrow \quad N'\cdot T = -T'\cdot N$$

$$= -\kappa\, N\cdot N = -\kappa$$

$$N\cdot B = 0, \quad N'\cdot B = -B'\cdot N = \tau$$

Applications

Lemma

$\beta : I \to \mathbb{R}^3$, unit speed $\kappa(s) > 0$ $\forall s$
then β is planar iff $\tau \equiv 0$

Proof

Assume β is planar (plane defined by point and normal vector)

point normal vector

$\forall s$, $(\beta(s) - \bar{p}) \cdot \bar{\eta} \equiv 0$

$\beta'(s) \cdot \bar{\eta} = 0$ $\beta''(s) \cdot \bar{n} \equiv 0$

$T(s) \cdot \eta = 0$

$\kappa(s) N(s) \bar{\eta} = 0$

$T(s)$, $N(s) \perp \eta$ $\forall s$

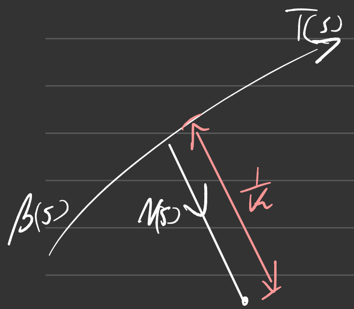
Hence $B(s) = \pm \eta$. In particular B is constant hence $(0,0,0) = B' = -\tau N$

$\Rightarrow \tau = 0$ (otherwise is exercise)

Lemma 2

β is a unit speed curve $\kappa = \text{constant} > 0$.
Then β forms part of a unit circle of radius $\frac{1}{\kappa}$

Proof



Define curve

$$\gamma'(s) = \beta'(s) + \frac{1}{\kappa} N'(s)$$

$$= T(s) + \frac{1}{\kappa}\left[-\kappa T(s) + \tau(s) B(s)\right]$$

$$= \left(T(s) - T(s)\right) + T(s)\, B(s)$$

$$= 0 + 0 \quad \text{since } \tau = 0$$

center

So $\gamma(s) = \bar{c}$, constant

$$\bar{c} = \beta(s) - \frac{1}{\kappa} N(s)$$

$$|\beta(s) - \bar{c}| = \left|\beta(s) - \beta(s) \frac{1}{\kappa} N(s)\right|$$

$$= \frac{1}{\kappa} \quad \text{(radius)}$$

Calculus Review

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\quad (n, m \in \{1, 2, 3\})$

We say f is smooth (differentiable) at $a \in \mathbb{R}^n$, if there is a linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ so that

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - L(h)|}{|h|} \qquad (h \in \mathbb{R}^n)$$

The linear map L is unique and called the derivative of f at a

<u>Notation</u>: $L = df_a = Df_a = f_*$

In the case when $m = 1$, then for any $\bar{v} \in \mathbb{R}^n$

$df_a(\bar{v})$ is called the directional derivative of f at a in the direction $\bar{v}$

Suppose $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ is the standard basis for $\mathbb{R}^n$ $(e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0) \dots)$

$df_a(\bar{e}_i) = \frac{\partial f}{\partial x_i}(a)$ $\quad i^{th}$ partial derivative

Thus

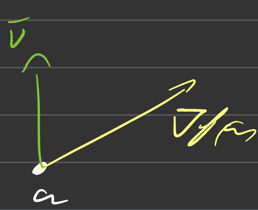
$$df_a(\bar{v}) = df_a\left(\sum_{i=1}^{n} v_i \bar{e}_i\right)$$

$$= \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(a)$$

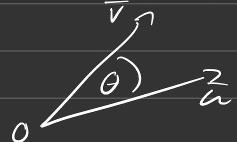
$$= (v_1, \ldots, v_n) \cdot \left(\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a)\right)$$

$$= \bar{v} \cdot \underbrace{\nabla f(a)}_{\text{gradient of } f \text{ at } a}$$

$$\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a)\right)$$



$$\bar{u} \cdot \bar{v} = |\bar{u}||\bar{v}|\cos\theta$$



Assume for convenience $|\bar{v}| = 1$

$$df_a(\bar{v}) = |\bar{v}||\nabla f(a)|\cos\theta$$

$$= |\nabla f(a)|\cos\theta \quad , \text{since } \bar{v} = 1$$

This is maximal when $\theta = 0$, ie when $\bar{v}$ points in the same direction as $\nabla f(a)$

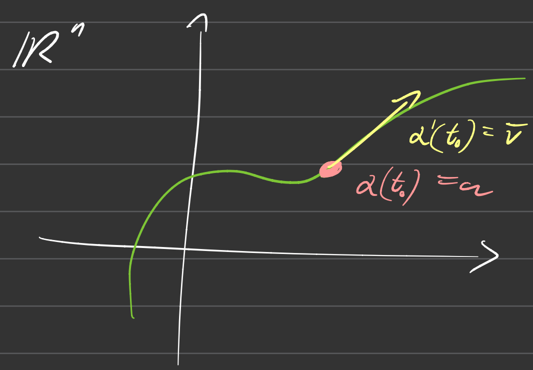
Back to $m \geq 1$, $f(x_1,\dots,x_n) = (f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n))$

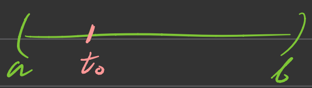
Easy to show that with respect to standard bases

$$\text{Matrix}(df_a) := \boxed{\begin{matrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{matrix}}$$

We say f is differentiable on its domain if df_a exists $\forall\, a \in \text{Domain } f$

We will normally only work with functions which are <u>infinitely differentiable</u> ie all partial derivatives of all orders exist





Consider $\quad f(\alpha(t)) \quad , \; t \in I$

$$\boxed{df_a(\bar{v}) = \frac{d}{dt}\bigg|_{t=t_0} f(\alpha(t))}$$

$$= f'(\alpha(t_0)) \cdot \alpha'(t)$$

$$= \nabla f(\alpha(t_0)) \cdot \alpha'(t)$$

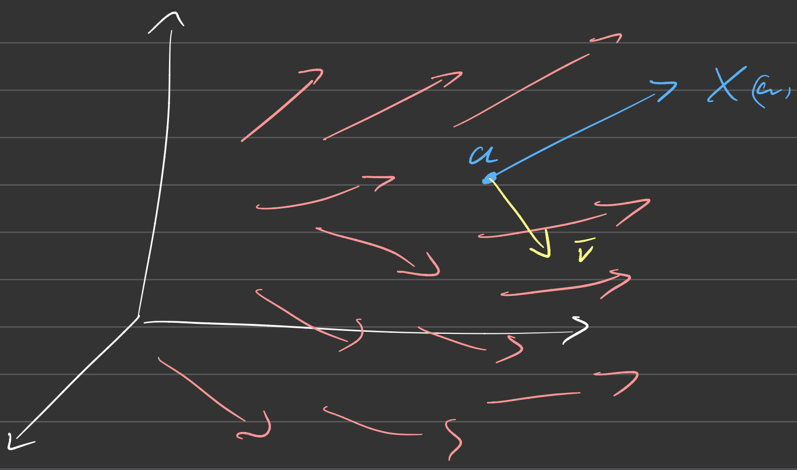
$$= \nabla f(a) \cdot \bar{v}$$

Special Case: Vector fields

A vector field, X, on $\mathbb{R}^3$ is a smooth map

$$X : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

which associates to each point $a \in \mathbb{R}^3$, a vector $X(a) = (X_1(a), X_2(a), X_3(a))$



Define the (covariant) derivative of X at a in the direction $\bar{v}$

$$\nabla_{\vec{v}} X(a) := \frac{d}{dt}\Big|_{t=0} X(a + t\bar{v})$$

Properties

X, Y vector fields on $\mathbb{R}^3$

$\bar{w}, \bar{v} \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$ constant

$f : \mathbb{R}^3 \to \mathbb{R}$ smooth, $a \in \mathbb{R}^3$

- $\nabla_{\bar{v}} (\lambda X + Y)(a) = \lambda \nabla_{\bar{v}} X(a) + \nabla_{\bar{v}} Y(a)$

- $\nabla_{\lambda \bar{w} + \bar{v}} X(a) = \lambda \nabla_{\bar{w}} X(a) + \nabla_{\bar{v}} X(a)$

- $\nabla_{\bar{v}} fX(a) = f(a) \nabla_{\bar{v}} X(a) + df_a(\bar{v}) \cdot X(a)$

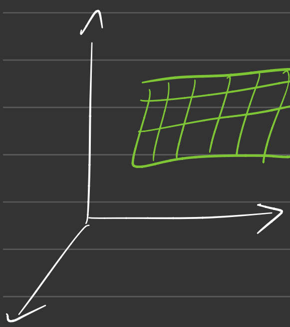
$X : \mathbb{R}^3 \to \mathbb{R}^3$
$f : \mathbb{R}^3 \to \mathbb{R}$

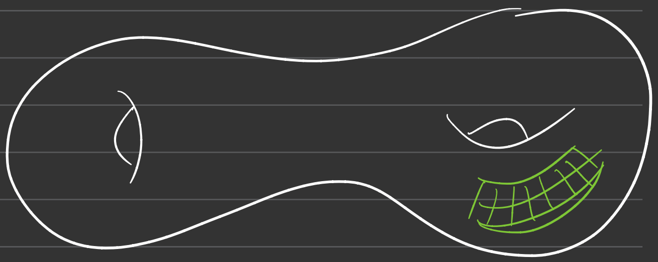
$fX : \mathbb{R}^3 \to \mathbb{R}^3$
$p \longmapsto \underbrace{f(p)}_{\text{scalar}} X(p)$

- $\nabla_{\bar{v}} (X \cdot Y)(a) = X(a) \cdot \nabla_{\bar{v}} Y(a) + (\nabla_{\bar{v}} X(a)) \cdot Y(a)$

Surfaces in $\mathbb{R}^3$

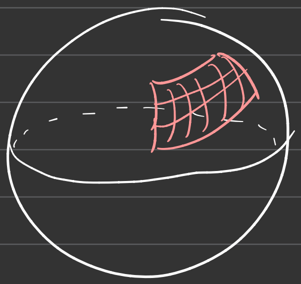
Idea: Roughly, a surface, $M \subset \mathbb{R}^3$, is a subset of $\mathbb{R}^3$ which locally resembles $\mathbb{R}^2$





Example

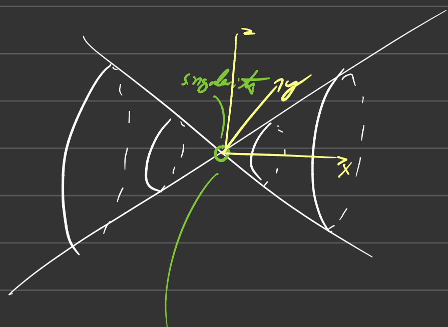
$$S^2 := \{ x \in \mathbb{R}^3 : |x| = 1 \}$$



unit sphere

Non Example

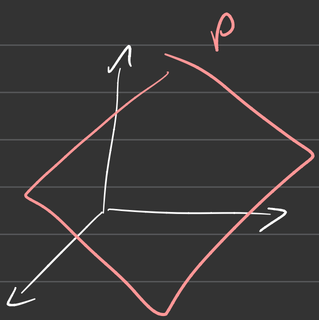
$\{(x,y,z) \in \mathbb{R}^2 \mid x^2 - y^2 - z^2 = 1\}$

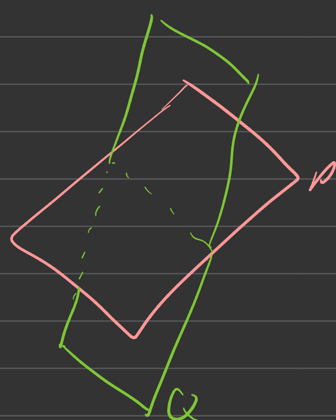


No neighbourhood of origin resembles $\mathbb{R}^3$

Example

Any 2-dim plane in $\mathbb{R}^3$





Making this Precise

Review:

$\bar{x}, \bar{y} \in \mathbb{R}^n$,

Euclidean dot product, $\bar{x} \cdot \bar{x} = x_1 y_1 + \dots + x_n y_n$

Euclidean Norm, $|\bar{x}| = \sqrt{\bar{x} \cdot \bar{x}}$

Euclidean distance $d_{Euc}(\bar{x}, \bar{y}) = |\bar{x} - \bar{y}|$

$B_{Euc}(x, \varepsilon) = \{ y \in \mathbb{R}^n \mid d_{Euc}(\bar{x}, \bar{y}) < \varepsilon \}$

$U \subset \mathbb{R}^n$ is open in $\mathbb{R}^n$ if $\forall\ x \in U$
$\exists\ \varepsilon > 0$ so that

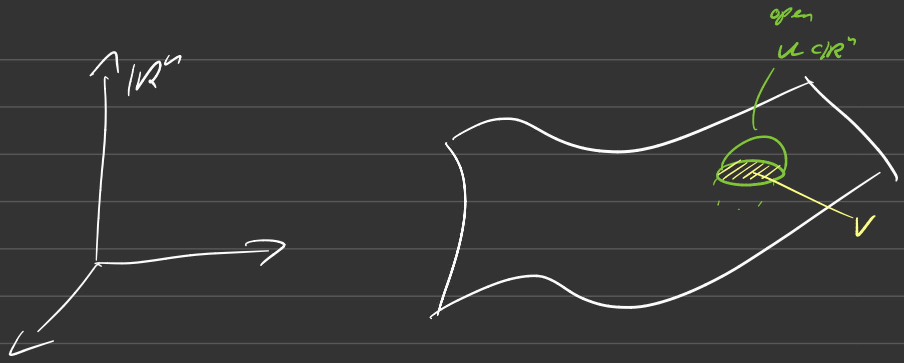
$B_{Euc}(x, \varepsilon) \subset U$

$K \subset \mathbb{R}^n$ arbitrary subset

$V \subset K$ is an open subset of K if

$V = U \cap K$ for some U

an open subset of $\mathbb{R}^n$



$K_1 \subset \mathbb{R}^{n_1}$, $K_2 \subset \mathbb{R}^{n_2}$ arbitrary subsets

$f: K_1 \longrightarrow K_2$, a function, is continuous if for any $V \subset K_2$ (open in K_2), the pre image $f^{-1}(V)$ is open in K_1

- If $f: K_1 \longrightarrow K_2$ is continuous, bijective and has a continuous inverse, then we say f is a homeomorphism (we say K_1 and K_2 are homeomorphic

$$K_1 \cong K_2$$

~~Example~~

$$f: \mathbb{R} \longrightarrow (0, \infty)$$
$$x \longmapsto e^x$$

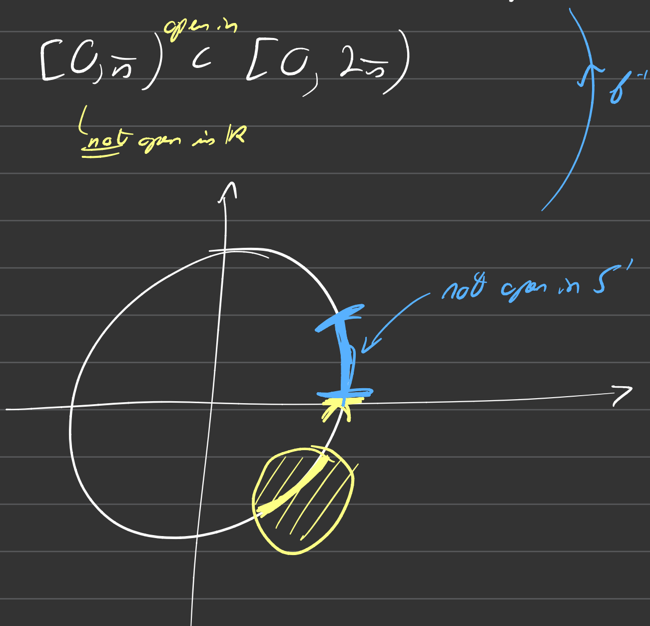
$$\mathbb{R} \cong (0, \infty)$$

Non Example

$$[0, 2\pi) \longrightarrow S^1 \subset \mathbb{R}^2$$
$$t \longmapsto (\cos t, \sin t)$$

$$[0,\pi) \overset{\text{open in}}{\subset} [0, 2\pi)$$

not open in $\mathbb{R}$

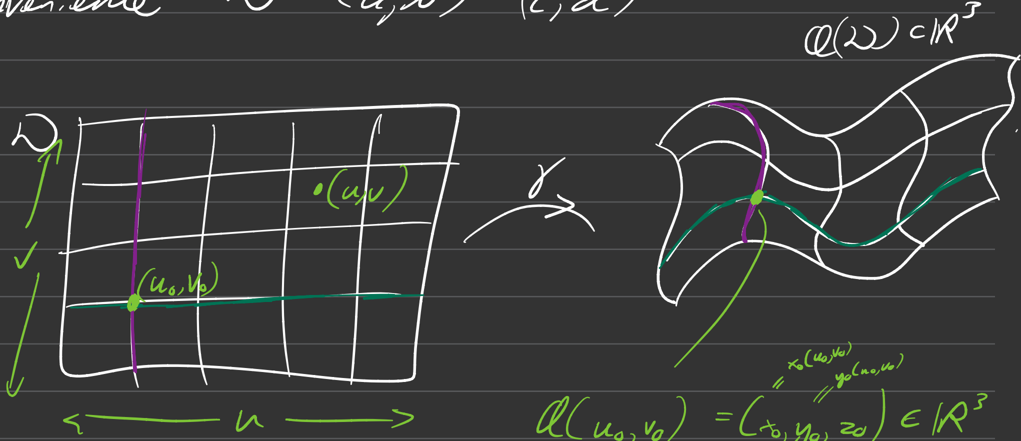


Defining a surface

• Local coordinates

$\emptyset \neq \mathcal{D} \subset \mathbb{R}^2$ open (connected) subset

Convenience $\mathcal{D} = (a,b) \times (c,d)$



$$\sigma(u_0,v_0) = (\underbrace{x_0}_{=x_0(u_0,v_0)}, \underbrace{y_0}_{=y_0(u_0,v_0)}, \underbrace{z_0}_{=z_0(u_0,v_0)}) \in \mathbb{R}^3$$

$\sigma : \mathcal{D} \to \mathbb{R}^3$

Smooth injective map satisfying

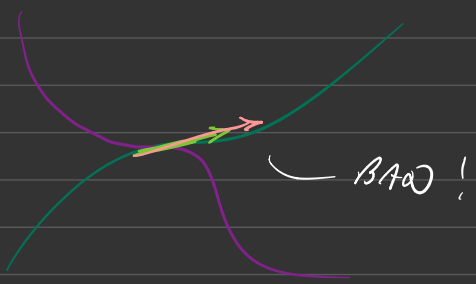
- regularity
- properness

Regularity:

$\mathbf{x}_u(u_0, v_0)$ = velocity vector of $u \mapsto \mathbf{x}(u, v_0)$

$\mathbf{x}_v(u_0, v_0)$ = ... $v \mapsto \mathbf{x}(u_0, v)$

Do not want a non transverse intersection of these paths at (u_0, v_0)



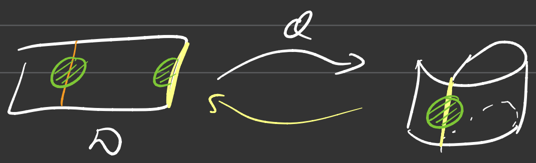
To avoid this insist that

$$|\mathbf{x}_u(u_0, v_0) \times \mathbf{x}_v(u_0, v_0)| > 0$$

Proper:

$\mathbf{x}^{-1}$ is continuous on all of $\mathbf{x}(D)$

This is to avoid the following problem



Recall

a coordinate patch is a smooth injective map

$$\alpha : D \longrightarrow \mathbb{R}^2$$
$$(u,v) \longmapsto \alpha(u,v)$$

($D \neq \emptyset$ open, connected region of $\mathbb{R}^2$)

which satisfies the following conditions

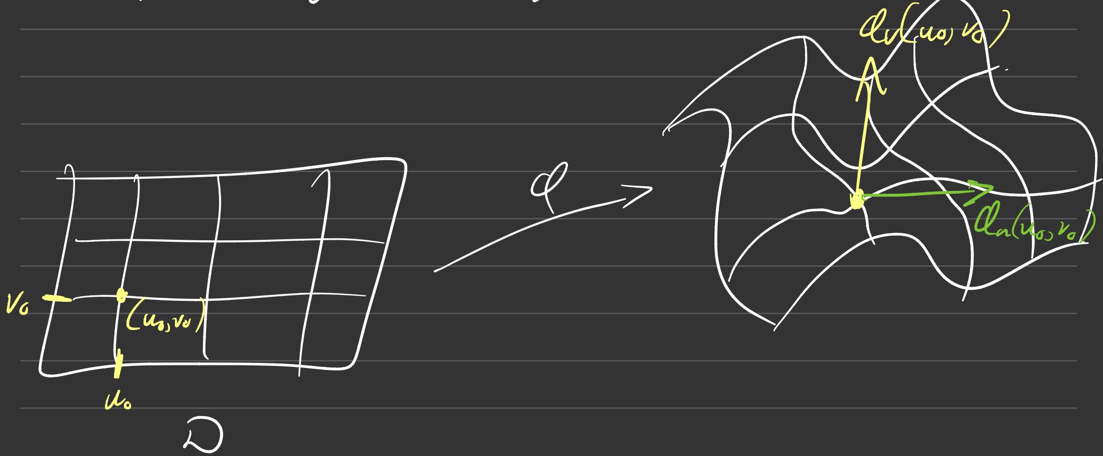
- Regularity

$\forall (u_0,v_0) \in D, \ \| \alpha_u(u_0,v_0) \times \alpha_v(u_0,v_0) \| > 0$

- Proper (ness)

α has continuous inverse $\alpha^{-1} : \operatorname{Im}\alpha \to D$

The image $\operatorname{im}\alpha = \alpha(D)$ is an example of a surface



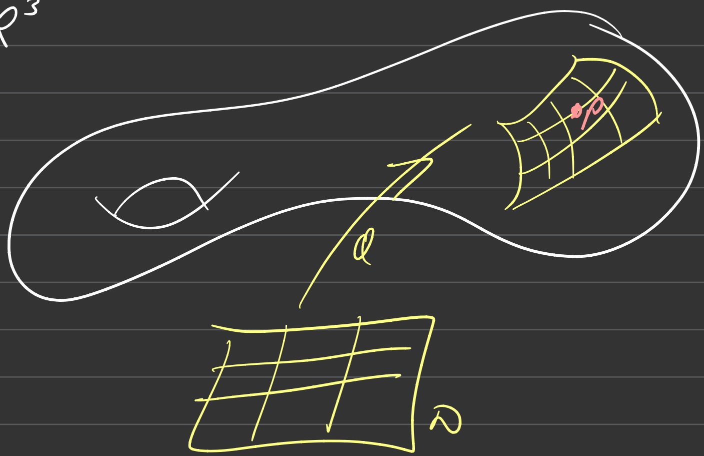
Definition

A surface, M, in $\mathbb{R}^3$, is a subset of $\mathbb{R}^3$ which satisfies the condition that $\forall\, p \in M$ there is an open set $\mathcal{O}$ of M and a coordinate patch

$$\alpha : D \longrightarrow M \subset \mathbb{R}^3 \text{ so that}$$

$$p \in \mathcal{O} \subset \operatorname{Im} \alpha \subset M$$

$M \subset \mathbb{R}^3$



Examples

(0) The image of a coordinate patch is always a surface

(1) Any 2-dim plane in $\mathbb{R}^3$

$P \subset \mathbb{R}^3$, $p \in P$, $\bar{s}, \bar{t} \in \mathbb{R}^3$

and $\operatorname{span}\{\bar{s}, \bar{t}\} = P - p$

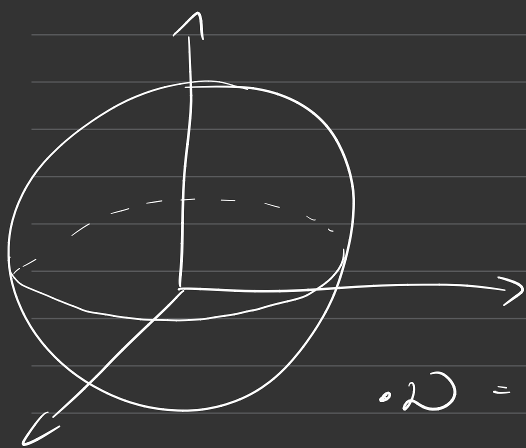
Then $\alpha : \mathbb{R}^2 \longrightarrow P$

$(u,v) \longmapsto p + u\bar{s} + v\bar{t}$

is a coordinate patch covering all P

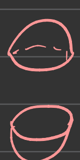
(2) 2-dim sphere

$S^2 = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \}$



Lots of coverings of S^2 by co-ordinate patches

$\cdot\, D = D^2 = \{ (u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1 \}$



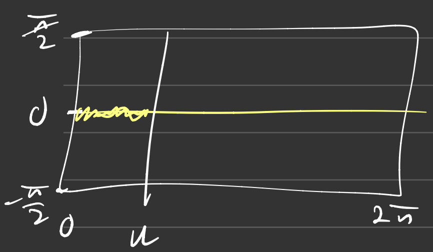
$\alpha_3^{\pm} : D \longrightarrow \mathbb{R}^3$

$(u,v) \longmapsto (u, v, \pm\sqrt{1-u^2-v^2})$

$$\mathcal{U}_2^{\pm} : D \longrightarrow \mathbb{R}^3$$
$$(u,v) \longmapsto (u, \pm\sqrt{1-u^2-v^2}, v)$$

$$\mathcal{U}_1^{\pm} \quad . \quad . \quad .$$

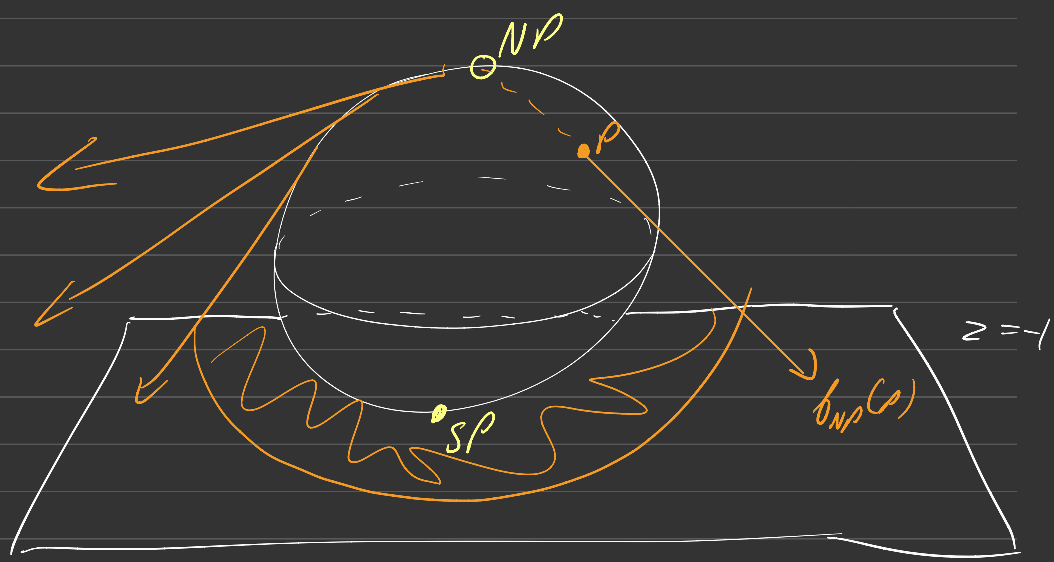
- $D = (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \xrightarrow{\mathcal{U}} \mathbb{R}^3$



$$(u,v) \longmapsto \begin{bmatrix} \cos v \cos u \\ \cos v \sin u \\ \sin v \end{bmatrix}$$

2 such patches configured so that the missing seams do not intersect covers S^2

Stereographic Coordinates



Consider $S^2 \setminus \{NP\}$

$$\varphi_{NP} : \mathbb{R}^2 \longrightarrow S^2 \setminus \{NP\} \subset \mathbb{R}^2$$
$$\bar{u} \longmapsto f_{NP}^{-1}(\bar{u})$$

Similarly define

$\varphi_{SP} : S^2 \setminus \{SP\} \longrightarrow \mathbb{R}^3$ to cover S^2

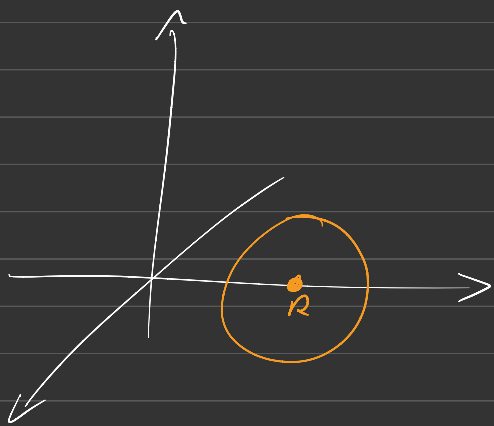
Examples of Surfaces

- Planes in $\mathbb{R}^3$

- $S^2(r) = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = r^2\}$

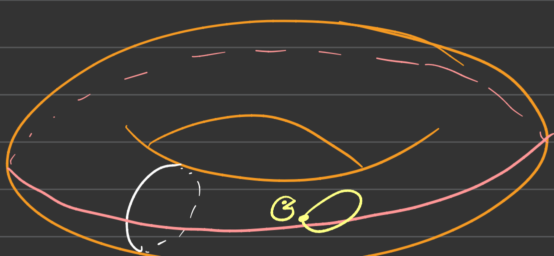
saw several examples of coverings of S^2 with co-ordinate patches

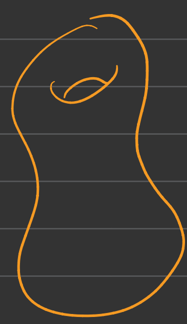
- Torus (T^2)



$0 < r < R$
circle centered at $(R,0)$ on the y-z plane

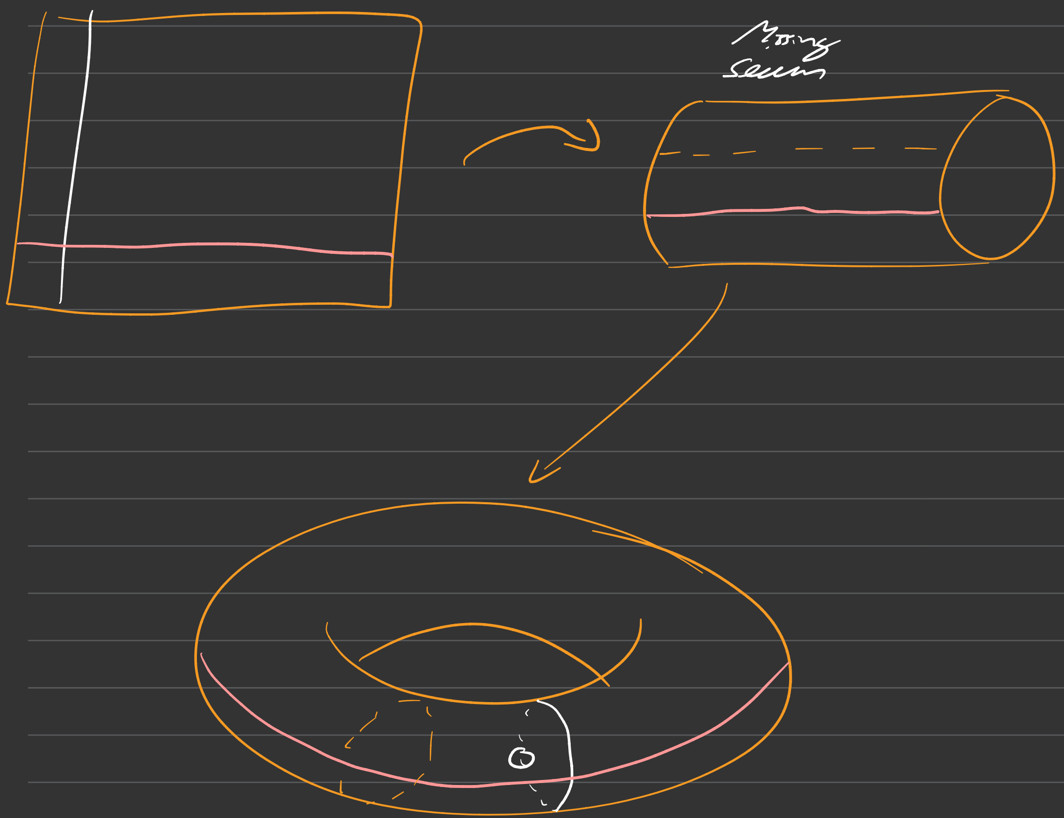
Trace out torus by rotating circle around the z-axis





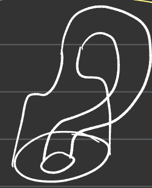
Exercise

Demonstrate that T^2 is a surface. In particular, find a parametrisation (co-ord patch) which covers "most" of T^2



Aside: In this course we deal with surfaces which are embedded in $\mathbb{R}^3$. However there is a more general definition of surface which allows for objects which are are not "embeddable" in $\mathbb{R}^3$

Classification of Surfaces

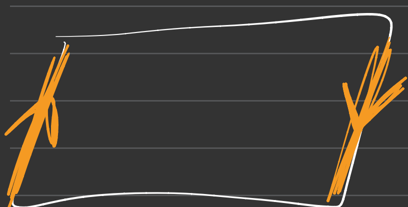


• Example (in $\mathbb{R}^3$ first)

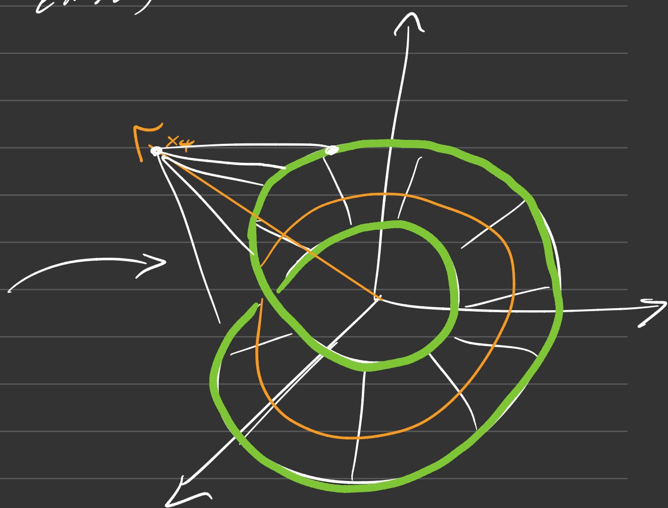
Mobius band

$\partial M\ddot{o} = S^1$ circle

Boundary



Rectangle



2 non embeddable in $\mathbb{R}^3$ examples

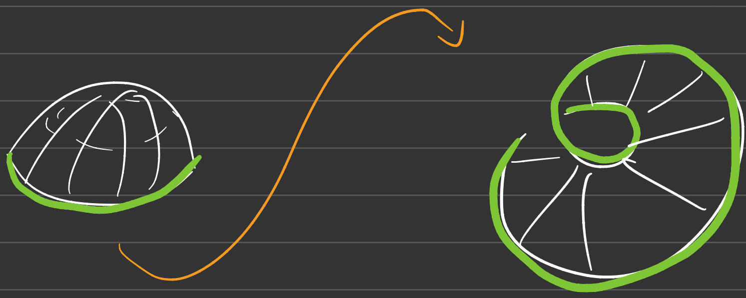
Consider gluing a disk D^2

$D^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, $\partial D^2 = S^1$

Challenge

Glue D^2 to $M_\ddot{o}$ by identifying boundaries

D^2



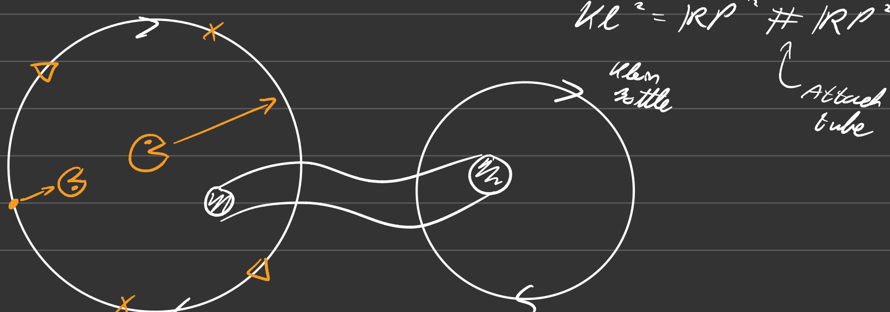
$$\partial D^n = S^{n-1}$$

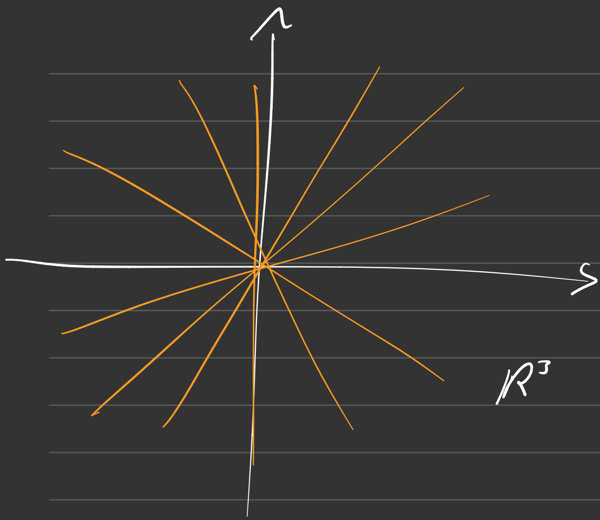
In $\mathbb{R}^3$ this results in self intersection and the result is not a surface in $\mathbb{R}^3$ but it is possible in $\mathbb{R}^4$

It is called $\mathbb{RP}^2$ (2-dim) real projective space

Alternative description of $\mathbb{R}^2$

$Kl^2 = \mathbb{RP}^2 \# \mathbb{RP}^2$



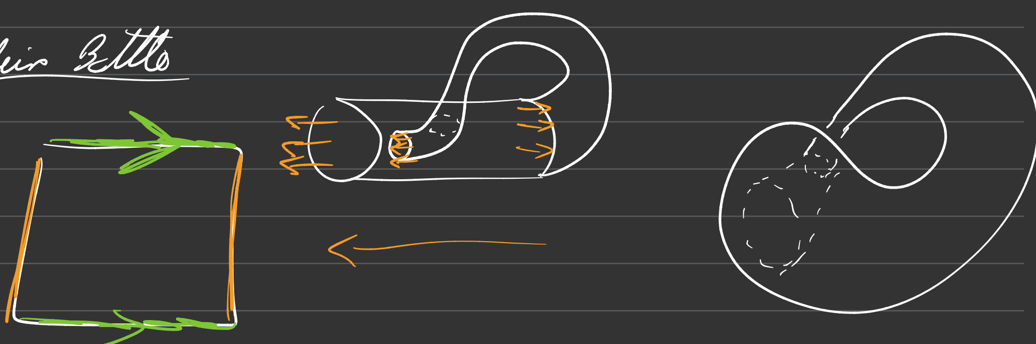


$\mathbb{RP}^2$ = 1-dim subspaces of $\mathbb{R}^3$

Kl^2 (Klein bottle) is obtained by gluing Mö to another Mö along boundary. Once again, need to be in $\mathbb{R}^4$

<u>Torus</u>

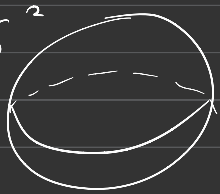
<u>Klein Bottle</u>

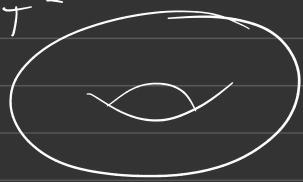


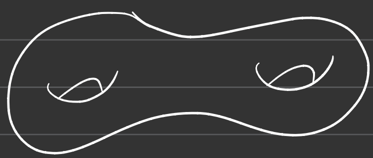
Examples of Surfaces

• Planes in $\mathbb{R}^3$

• S^2 $\quad$ T^2

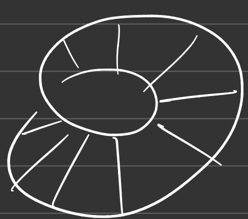






$T^2 \# T^2$

• Möbius strip $\subset \mathbb{R}^3$ (without border)



☆ Non-orientable closed surfaces

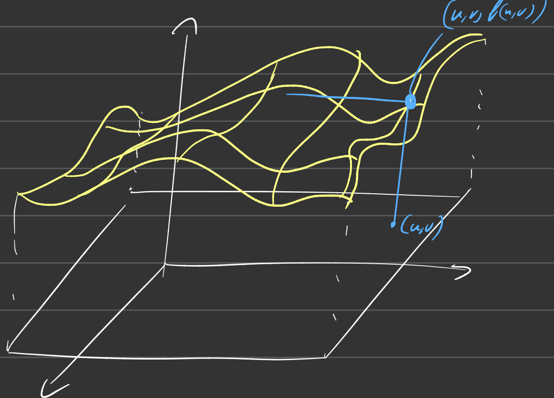
$\mathbb{RP}^2$, Kl^2 (Klein bottle) $\not\subset \mathbb{R}^3$

• Monge Surface

M is the graph of a smooth function of a smooth function

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\emptyset \neq D$ open connected region in $\mathbb{R}^2$



$\varphi(u,v) = (u,v,f(u,v))$

Exercise: Verify that φ is a co-ord patch

• $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ smooth function

A critical point of f is a point $p \in \mathbb{R}^3$ so that

$$\nabla f(p) = (0,0,0)$$

so $\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0$

A regular point of f is a point p which is not critical

A critical value of f is an element $c \in \mathbb{R}$ for which $f^{-1}(c)$ contains at least one critical point.

If $f^{-1}(c)$ contains only regular points we call c a regular value

$f(x,y,z) = c$ $\quad \nabla f(c) \neq 0$

$f^{-1}(c) = \{(x,y,z) \mid f(x,y,z) = c\}$

If c is a regular value, $f^{-1}(c)$ is called a regular level set

Theorem

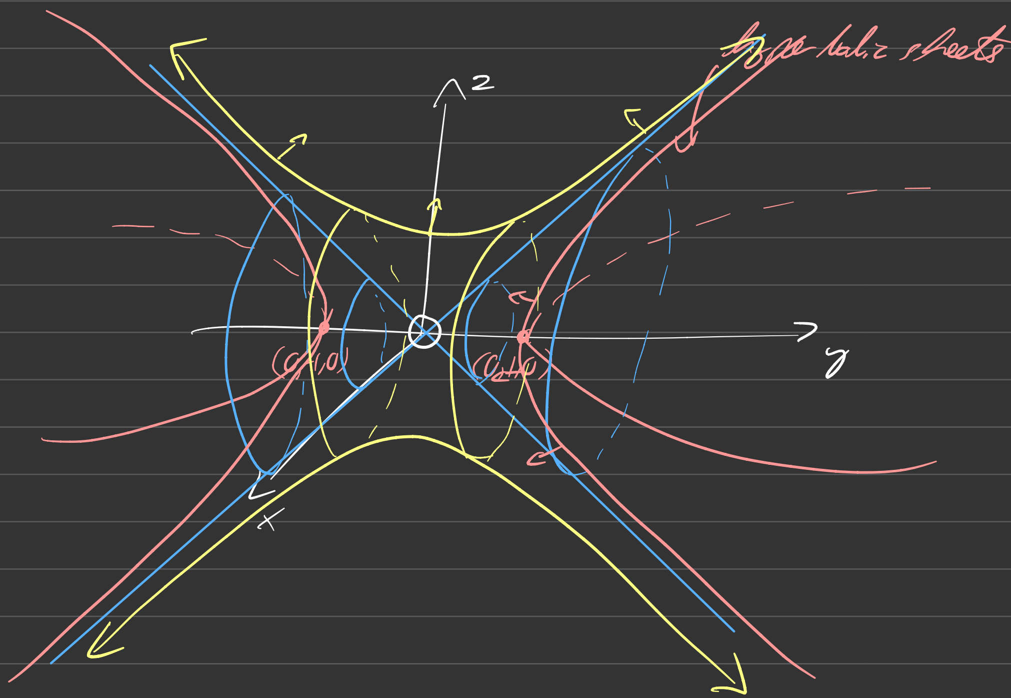
A non-empty regular level set of f is allways a surface in $\mathbb{R}^3$

Proof Later (key ingredient: implicit function theorem)

• $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

eg $f(x,y,z) = x^2 - y^2 + z^2$

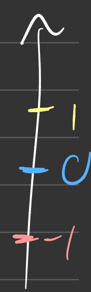
$\nabla f = (2x, -2y, 2z)$ only crit point $= (0,0,0)$



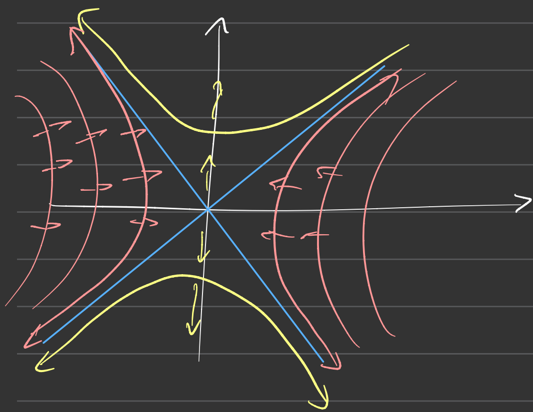
$f^{-1}(-1)^{\text{reg}}$ $x^2 - y^2 + z^2 = -1$

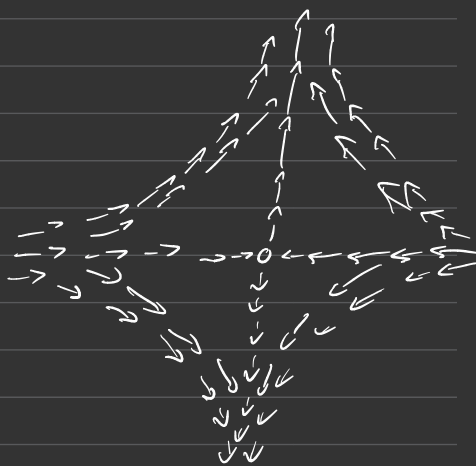
$f^{-1}(0)^{\text{crit}}$ $x^2 - y^2 + z^2 = 0$

$f^{-1}(1)^{\text{reg}}$ $x^2 - y^2 + z^2 = 1$



2-dim slice of y-z plane





Using the Regular level set theorem it is immediate that

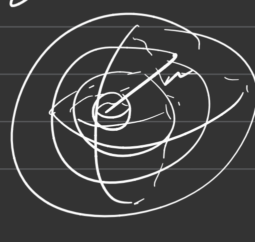
$f^{-1}(c)$ is a surface whenever $c \neq 0$

- $f : \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x,y,z) = x^2+y^2+z^2$

$S^2(r) = f^{-1}(r^2) \quad r>0$

This is a surface by RLS theorem since $f^{-1}(0)$ is the only non-reg set

$\nabla f = (2x, 2y, 2z)$

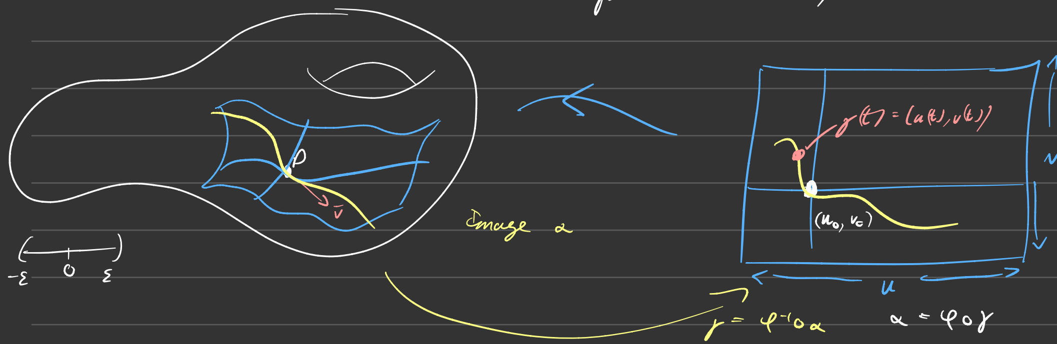


$p \in M \subset \mathbb{R}^3$ is a surface

$\alpha : (-\varepsilon, \varepsilon) \longrightarrow M \quad , \varepsilon > 0$

$\alpha(0) = p \qquad \alpha'(0) =: \bar{v}$

$p = \varphi(u_0, v_0)$



Any vector $\bar{v}$ arising in this way is said to be tangent to M at p

The set of all all such vectors is called the tangent space to M at p, denoted T_pM

[Exercise: Show that T_pM is a vector space

$\alpha'(0) = (\varphi \circ \gamma)'(0)$

$$= \frac{d}{dt}\Big|_{t=0} \varphi(u(t), v(t))$$

$$= \varphi_u(u_0, v_0) \cdot u'(0) + \varphi_v(u_0, v_0) \cdot v'(0) \qquad \begin{aligned} u(0) &= u_0 \\ v(0) &= v_0 \end{aligned}$$

$$=$$

Excercise:

Prove that

$$T_pM = \operatorname{span}\{\varphi_u(u_0, v_0), \varphi_v(u_0, v_0)\}$$

$$\cong \mathbb{R}^2$$

linearly independent by definition

Brief aside

$M \subset \mathbb{R}^3$, surface

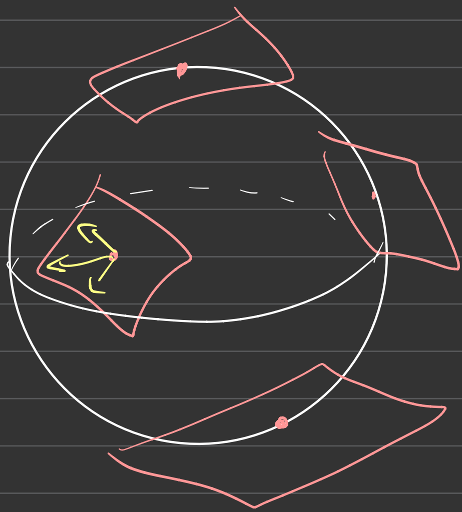
$$TM := \{(p, \bar{v}) \mid p \in M, \bar{v} \in T_pM\}$$

$$:= \bigsqcup_{p \in M} T_pM \quad \text{, the tangent bundle to } M$$

TM is a 4-dim space

Example

$\mathbb{R}^2 \cong$



S^2 – base

Each $T_pS^2 = \mathbb{R}^2$ – fibres

TS^2 is an example of a fibre bundle

As a space, surely TS^2 is just the same as $S^2 \times \mathbb{R}^2$? NO!

$$TS^2 \not\cong S^2 \times \mathbb{R}^2$$

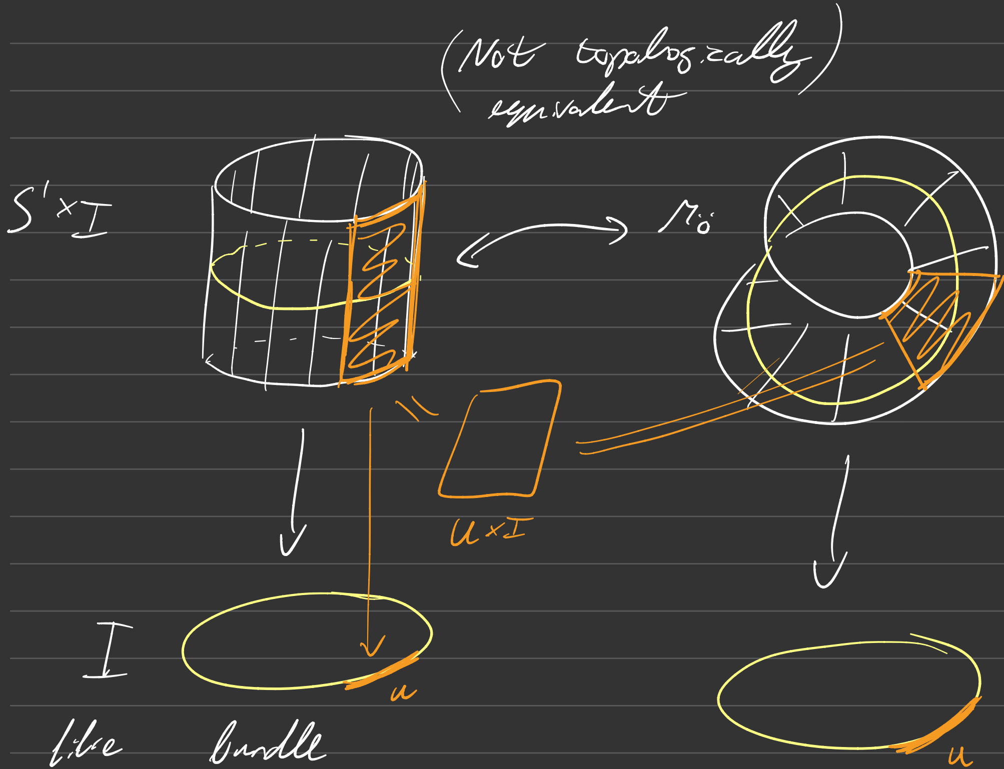
Why not? Because TS^2 is <u>twisted</u>!

Simple Examples

2 Bundles

Base = S^1 (circle)

Fibre = $[-1, 1]$ interval



$$S^n = \{ \bar{x} \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1 \}$$

TS^n = tangent bundle where each tangent space

$$T_p S^n \cong \mathbb{R}^n$$

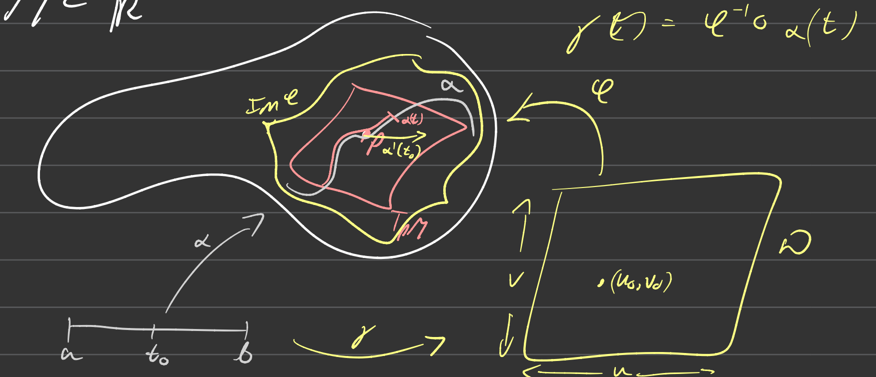
Theorem (Adams)

$$TS^n \cong S^n \times \mathbb{R}^n \iff n = 1, 3 \text{ or } 7$$

$p = \alpha(t_0) = \varphi(u_0, v_0)$

$\gamma(t) = \varphi^{-1} \circ \alpha(t)$

$M \subset \mathbb{R}^3$



$$\text{Length } \alpha = \int_a^b |\alpha'(t)|\,dt$$

Length along α from a to t

$$=: s(t) = \int_a^t |\alpha'(l)|\,dl$$

Assume $\operatorname{Im}\alpha \subset \operatorname{Im}\varphi$ for some co-ord patch

$$\varphi : D \longrightarrow M \subset \mathbb{R}^3$$

$$= \left(\frac{ds}{dt}\right)^2 = s'(t)^2 = |\alpha'(t)|^2 = \alpha'(t)\cdot\alpha'(t)$$

Note $\alpha'(t) = \varphi_u \dfrac{du}{dt} + \varphi_v \dfrac{dv}{dt}$

$$\alpha'(t)\cdot\alpha'(t) = \varphi_u\cdot\varphi_u\left(\frac{du}{dt}\right)^2 + 2\varphi_u\cdot\varphi_v\frac{du}{dt}\frac{dv}{dt}$$
$$+ \varphi_v\cdot\varphi_v\left(\frac{dv}{dt}\right)^2$$

Common abbreviations

$$E = \varphi_u\cdot\varphi_u\,,\quad F = \varphi_u\cdot\varphi_v\,,\quad G = \varphi_v\cdot\varphi_v$$

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

This is known as (in patch coordinates) as the First Fundamental Form of M

$p \in M \subset \mathbb{R}^3$ surface

$\vec{v} \in T_pM :=$ tangent space to M at p

$\vec{v} = \alpha'(0)$ where $\alpha : (-\varepsilon, \varepsilon) \to M \subset \mathbb{R}^3$
is a curve satisfying $\alpha(0) = p$

$\varphi : \mathcal{D} \to M$ co-ord patch

$\varphi(u_0, v_0) = p$

$\{ \varphi_u(u_0, v_0), \varphi_v(u_0, v_0) \}$ are a basis

for T_pM

$$TM = \bigsqcup_{p \in M} T_pM \quad \text{tangent bundle to } M$$

A vector field on M is a smooth map

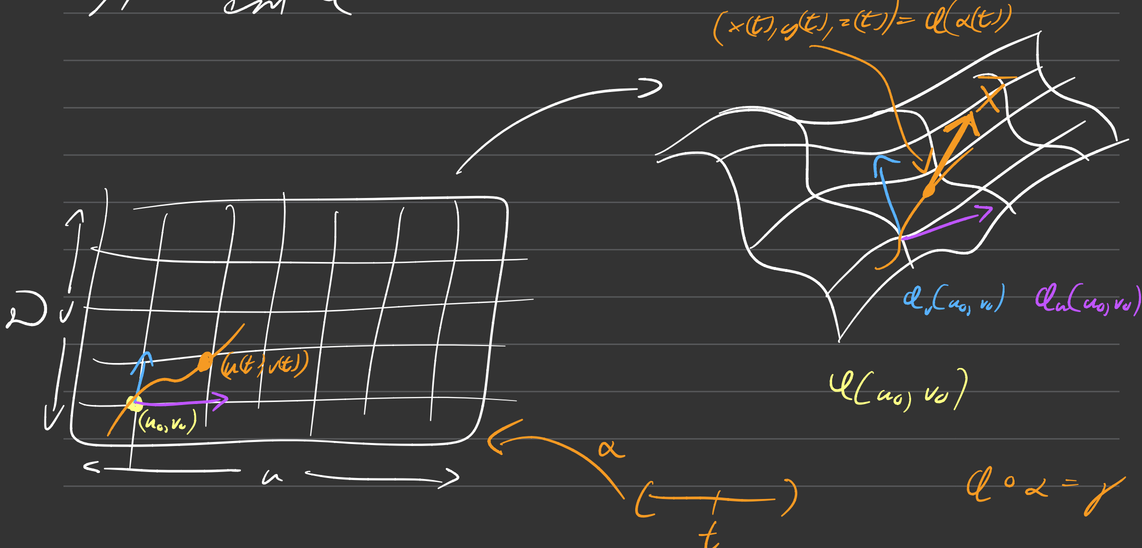
$$M \xrightarrow{X} \mathbb{R}^3$$

equipping each point $p \in M$ with a vector $X(p) \in \mathbb{R}^3$. In the case when $X(p) \in T_pM \ \forall p \in M$ we say X is a <u>tangent vector field</u>

Working in a co-ord patch

$\varphi : \Omega \to \mathbb{R}^3$ co-ord patch

$M = \operatorname{im} \varphi$



Curve in $\mathcal{D}$

$$\alpha(t) = (u(t), v(t))$$

$$\varphi \circ \alpha(t) = (x(t), y(t), z(t))$$

Length of $\gamma = \int_a^b |\gamma'(t)|\, dt$

$$\gamma'(t) = \frac{d}{dt}(\varphi \circ \alpha)(t)$$

$$= \varphi_u(\alpha(t))\, u'(t) + \varphi_v(\alpha(t))\, v'(t)$$

$$s(t) = \int_a^t |\gamma'(\ell)|\, d\ell$$

$$\frac{ds}{dt} = |\gamma'(t)|$$

$$\left(\frac{ds}{dt}\right)^2 = \gamma'(t) \cdot \gamma'(t)$$

$$= \left(\varphi_u(\alpha(t))\frac{du}{dt} + \varphi_v(\alpha(t))\frac{dv}{dt}\right) \cdot \left(\quad " \quad \right)$$

After simplification

$$ds^2 = \underbrace{\mathbf{x}_u \cdot \mathbf{x}_u \, du^2}_{= du \otimes du} + 2\mathbf{x}_u \cdot \mathbf{x}_v \underbrace{du\,dv}_{= \frac{1}{2}(du \otimes dv + dv \otimes du)}$$

$$+ \mathbf{x}_v \cdot \mathbf{x}_v \underbrace{dv^2}_{= dv \otimes dv}$$

Common notation

$$E = \mathbf{x}_u \cdot \mathbf{x}_u \,, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v \,, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v$$

$$\boxed{ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

First Fundamental Form

Aside

ds^2 is a smoothly varying inner product on D (associated with M via $\mathbf{x}$)

Suppose that $\bar{x}, \bar{y}$ are tangent vectors in T_pM

$$\bar{x} = x_u \mathbf{x}_u + x_v \mathbf{x}_v \qquad (\mathbf{x}_u = \mathbf{x}_u(u_0, v_0) \text{ etc})$$

$$\bar{y} = y_u \mathbf{x}_u + y_v \mathbf{x}_v$$

What is du?

$$du : \mathbb{R}^2 \longrightarrow \mathbb{R}$$ linear map
$$(a,b) \longmapsto a$$

$$dv : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(a,b) \longmapsto b$$

(bilinear)

$$du \otimes dv : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(\bar{a}, \bar{b}) \longmapsto du(\bar{a})dv(\bar{b})$$
$$= a_1 b_2$$

$\bar{a} = (a_1, a_2)$

$\bar{b} = (b_1, b_2)$

Not symmetric

Symmetric product of tensors

$$du\,dv = \frac{1}{2}(du \otimes dv + dv \otimes du)$$

$$\bar{x} = x_u \mathbf{e}_u + x_v \mathbf{e}_v \qquad \bar{y} = y_u \mathbf{e}_u + y_v \mathbf{e}_v$$

$$ds^2(\bar{x}, \bar{y}) = E\,du^2(\bar{x}, \bar{y}) + F\,du\,dv(\bar{x}, \bar{y})$$
$$+ G\,dv^2(\bar{x}, \bar{y})$$

$$= E x_u y_u + 2F\tfrac{1}{2}\left(du(\bar{x})\,dv(\bar{y}) + dv(\bar{x})\,du(\bar{y})\right) + G\, x_v y_v$$

$$= E x_u y_u + F(x_u y_v + x_v y_u) + G x_v y_v$$

In coordinates $\bar{x} = (x_u, x_v)$, $\bar{y} = (y_u, y_v)$

$$ds^2(\bar{x}, \bar{y}) = \underbrace{[x_u \quad x_v]}_{=\bar{x}} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \underbrace{\begin{bmatrix} y_u \\ y_v \end{bmatrix}}_{=(\bar{y})^T}$$

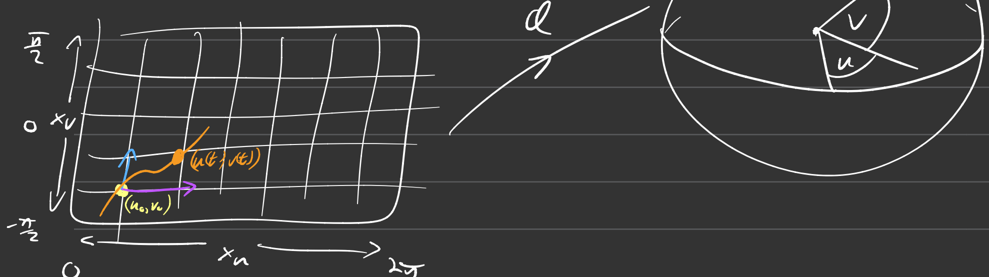
In $\mathbb{R}^2$, standard Euclidean dot product

$$\bar{x} \cdot \bar{y} = \bar{x} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\bar{y})^T$$

Example

$\varphi : \mathcal{D} \to \mathbb{R}^3$ coord patch

$M = \operatorname{im} \varphi$



$$\alpha(u,v) = \begin{bmatrix} \cos v \cdot \cos u \\ \cos v \ \sin u \\ \sin v \end{bmatrix} \in \mathbb{R}^3$$

$$\alpha_u = \begin{bmatrix} -\cos v \sin u \\ \cos v \ \cos u \\ 0 \end{bmatrix}$$

$$\alpha_v = \begin{bmatrix} -\sin v \cos u \\ -\sin v \ \sin u \\ \cos v \end{bmatrix}$$

$$E = \alpha_u \cdot \alpha_u = c^2v\, s^2u + c^2v\, c^2u$$

$$= c^2 v$$

$$F = cv\, su\, sv\, cu - cv\, su\, sv\, cu + 0$$

$$= 0$$

$$G = s^2v\, c^2u + s^2v\, s^2u + c^2v = s^2v + c^2v = 1$$

First fundamental form for S^2 in std spherical coords

$$\begin{bmatrix} \cos^2 v & 0 \\ 0 & 1 \end{bmatrix}$$

Last day

$S^2(R) \subset \mathbb{R}^3$

round sphere of radius R

Parameterised by spherical co-ords

$$(0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}^3$$

$u \qquad v$

$$\begin{bmatrix} \cos v \cos u \\ \cos v \sin u \\ \sin v \end{bmatrix}$$

After some work....

First fundamental form

$$ds^2 = R^2 dv^2 + R^2 \cos^2 v \, du^2$$

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

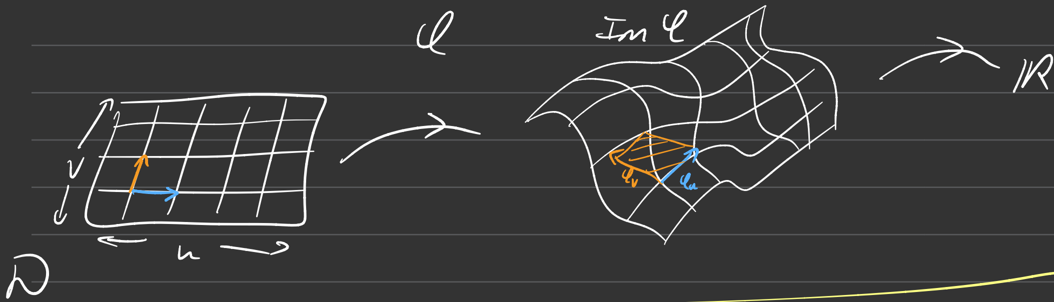
$$ds^2(\bar{x}, \bar{y}) = \bar{x} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \bar{y}^T$$

$$ds^2 = R^2 dv^2 + R^2 \cos^2 v \, du^2$$

$$\begin{bmatrix} R^2 \cos^2 v & 0 \\ 0 & R^2 \end{bmatrix}$$

Area (2-dim Volume)

Coordinate patch



$$\iint_{\operatorname{Im}\varphi} f \underbrace{dA}_{\text{area form/element}} := \int_c^d \int_a^b f \circ \varphi \underbrace{|\varphi_u \times \varphi_v|}_{\text{correction (Jacobian)}} \, du\, dv$$

Useful Tool

Lagrange: $|\varphi_u \times \varphi_v|^2 = EG - F^2$ (det of first fund form)

$E = R^2\cos^2 v$
$F = 0$
$G = R^2$

In the case of $S^2(R)$

$$\text{Area of Im } \ell = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} 1\sqrt{R^4\cos^2 v}\,du\,dv$$

$$= R^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos v\,du\,dv$$

$$= 2\pi R^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos v\,dv$$

$$= 2\pi R^2 \sin v\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 4\pi R^2$$

Mappings Between Surfaces

$M_1, M_2 \in \mathbb{R}^3$ are both surfaces

$f : M_1 \longrightarrow M_2$ map

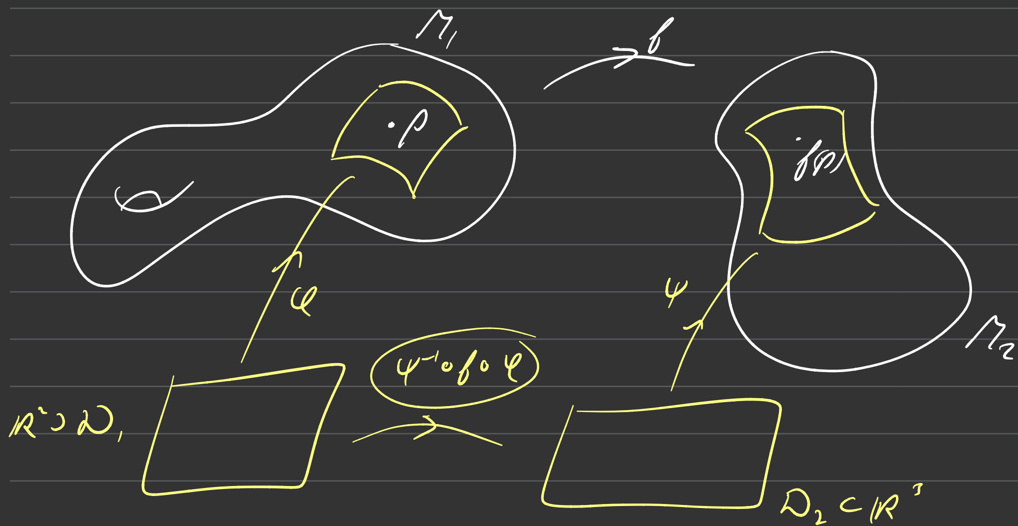
$p \in M_1$

We say f is smooth (differentiable) at p if there are coordinate patches

$\varphi : D_1 \longrightarrow M_1$, $p \in \operatorname{Im} \varphi$
$\psi : D_2 \longrightarrow M_2$ $f(p) \in \operatorname{Im} \psi$

$\psi^{-1} \circ f \circ \varphi : D_1 \longrightarrow D_2$

is smooth (in the usual sense)



If f is smooth $\forall p \in M_1$, we call $f: M_1 \to M_2$ a smooth map

If $f: M_1 \to M_2$ is smooth and bijective, with a smooth inverse, we call f a diffeomorphism from $M_1 \to M_2$

We in turn say M_1 is diffeomorphic to M_2 by

$$M_1 \cong M_2$$

Diffeomorphism is an equivalence relation

$M_1, M_2 \subset \mathbb{R}^3$ surfaces

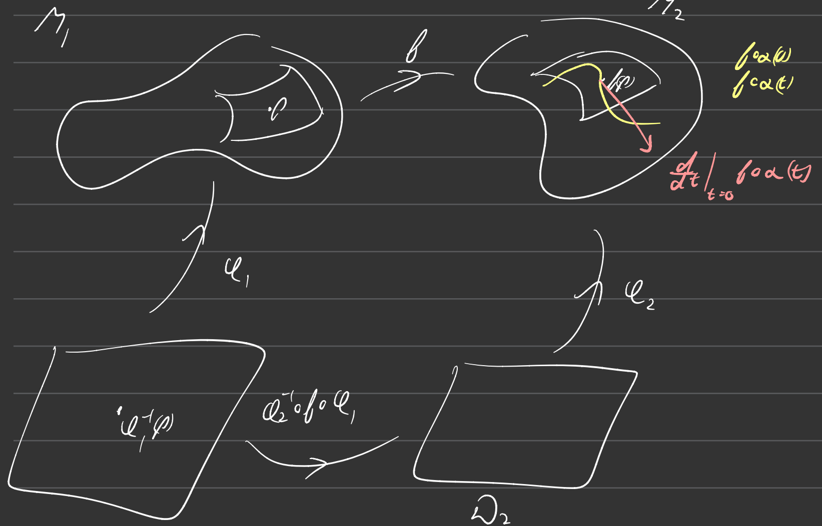
$f : M_1 \to M_2$ smooth map

$\forall p \in M_1$, there are co-ordinate patches

$\varphi_i : \Omega_i \to M_i$

$i \in \{1, 2\}$, satisfying

- $p \in \operatorname{Im}(\varphi_1)$, $f(p) \in \operatorname{Im} \varphi_2$
- $\varphi_2^{-1} \circ f \circ \varphi_1$ smooth at $\varphi_1^{-1}(p)$



Provided f smooth at p obtain the derivative of f at p,

denoted df_p (or Df_p, f_*, Tf_p)

$$df_p : T_pM_1 \longrightarrow T_{f(p)}M_2$$
$$\bar{v} \longmapsto df_p(\bar{v})$$

defined as follows

$\forall$ $\bar{v} \in T_pM$, let $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ be a smooth curve with $\alpha(0) = p$
$\alpha'(0) = \bar{v}$

and define

$$df_p(\bar{v}) = \frac{d}{dt}\Big|_{t=0} f(\alpha(t))$$

This is well defined for choice of α

Exerc. 81

Show $df_p : T_pM_1 \longrightarrow T_{f(p)}M_2$

is a linear map

Aside

df_p surjective $\forall p \Rightarrow f$ submersion

df_p injective $\forall p \Rightarrow f$ immersion

Vector fields (on Surfaces)

$M \subset \mathbb{R}^3$

A vector field, X, on M is a smooth map

$$X : M \longrightarrow \mathbb{R}^3$$

assigning a vector

$$X(p) = (X_1(p), X_2(p), X_3(p))$$

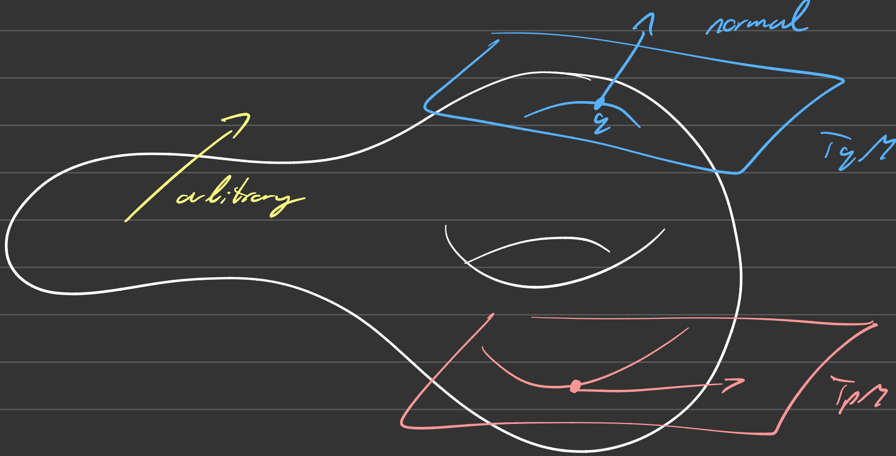
to each point $p \in M$

Definitions

(i) X is non-vanishing if $X(p) \neq \overline{0} \quad \forall p \in M$

(ii) X is a tangent vector field if $X(p) \in T_pM \quad \forall p \in M$

(iii) X is a normal vector field if $X(p)$ perpendicular to T_pM $\forall p \in M$



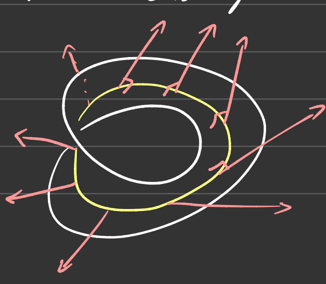
2 important questions

(1) does M admit a non-vanishing normal vector field (nvf)

(2) does M admit a non-vanishing tangent vector field (tvf)

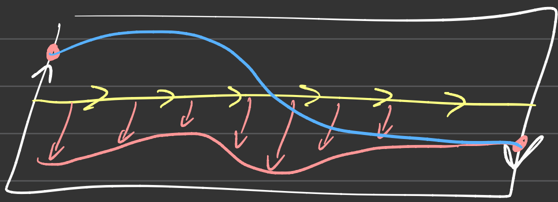
Answer (1), (2) Not always!

(1) non-example



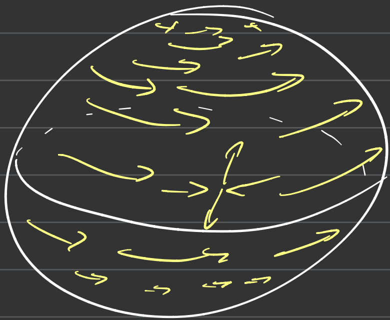
No non vanishing normal vector field to Möbius strip

Π



Any non vanishing vector field connects $a \neq 0$ to $b \neq 0$. By Intermediate value theorem, vec field has a zero $\amalg$

(2) Non-example: S^2



Hairy Ball Theorem

S^2 has no non vanishing tangent vector fields

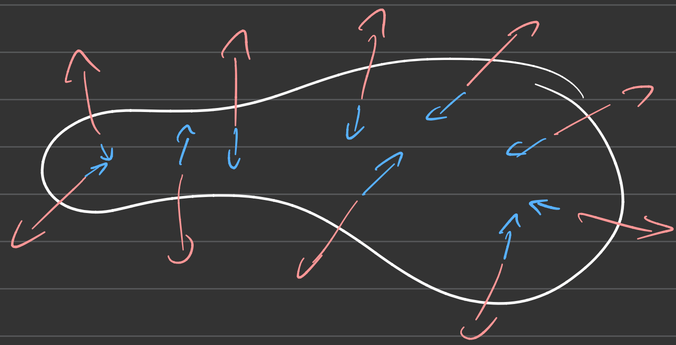
Definition

If $M \subset \mathbb{R}^3$ admits a non vanishing normal vector field, we say M is orientable

Suppose $X: M \to \mathbb{R}^3$ non-vanishing nvf

Define $U = \frac{X}{|X|}$ unit nvf

Obtain 2 opposite unit nvfs

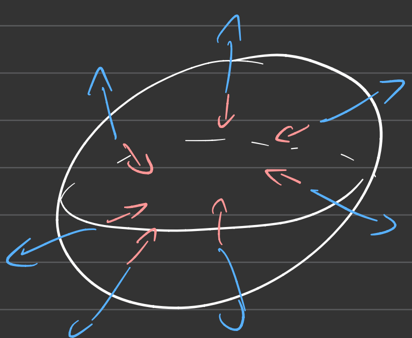


U unvf

$-U$

Orientable means "2-sided"

In this case $M \subset \mathbb{R}^3$ is orientable
2 choices of unit nvf. Each choice
is called an <u>orientation</u> on M



One orientation
(outside)

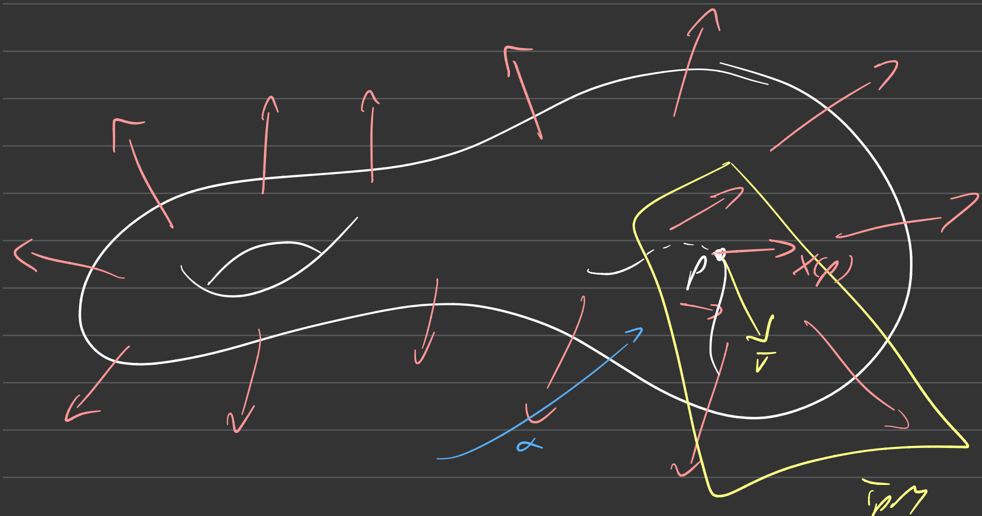
inside (opposite
orientation)

Covariant Differentiation

$M \subset \mathbb{R}^3$ surface

X a vector field on M

$p \in M$, $\bar{v} \in T_pM$ we wish to measure how X changes at p in the direction $\bar{v}$



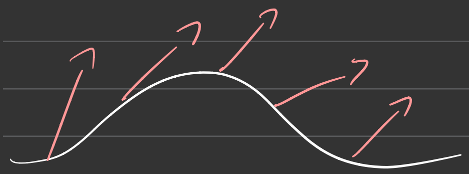
The covariant derivative of X at p in the direction $\bar{v}$ is denoted

$\nabla_{\bar{v}} X(p)$

and is defined as follows

Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a smooth curve satisfying

$$\alpha(0) = p \,, \quad \alpha'(0) = \bar{v}$$



$$X(\alpha(t)) =: X(t)$$

Then set

$$\nabla_{\bar{v}} X(p) = \left.\frac{d}{dt}\right|_{t=t_0} X(\alpha(t))$$

Equivalently, we may define / compute $\nabla_{\bar{v}} X(p)$ as follows

$$\bar{v} = v_1 \bar{e}_1 + v_2 \bar{e}_2 + v_3 \bar{e}_3$$

$$X(p) = \underbrace{X_1(p)}_{\text{smooth real valued func in } p} \bar{e}_1 + X_2(p) \bar{e}_2 + X_3(p) \bar{e}_3$$

$$\nabla_{\bar v} X(p) = \sum \underbrace{\bar v[X_i]}_{\text{directional derivative}}(p)$$

Each $\bar v[X_i](p) = \sum_{j=1}^{3} v_j \dfrac{\partial X_i}{\partial x_j}(p)$

Easy to deduce the following differentiation rules

$\bar u, \bar v \in T_pM$ $\qquad$ X, Y vec fields on M

- $\nabla_{a\bar u + b\bar v} X = a \nabla_{\bar u} X(p) + b \nabla_{\bar v} X(p)$
- $\nabla_{\bar u}(aX + bY) = a\nabla_{\bar u} X + b \nabla_{\bar u} Y$
- Suppose $f : M \to \mathbb{R}$ smooth func

$fX(p) := f(p)X(p)$

$$\nabla_{\bar v} fX = f(p)\nabla_{\bar v} X(p) + \underbrace{\bar v[f]}_{\substack{\text{directional derivative}\\ \text{of } f \text{ at } p \text{ in direction } \bar v}}(p)\, X(p)$$

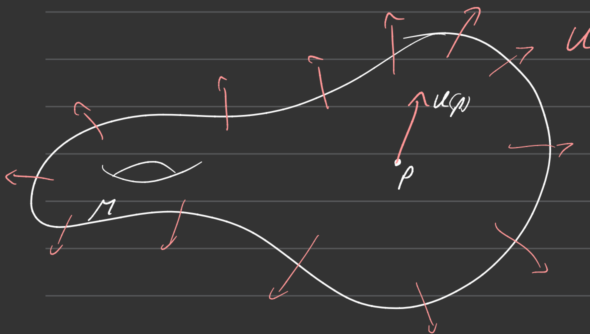
- $\nabla_{\bar v}(X \cdot Y)(p) = X(p) \cdot (\nabla_{\bar v} Y(p)) + (\nabla_{\bar v} X(p)) \cdot Y(p)$

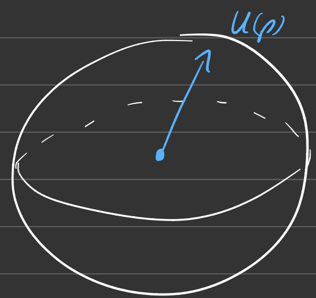
$\quad \| \quad$

$\bar v[X \cdot Y](p)$

Curvature On Surfaces

$M \subset \mathbb{R}^3$ orientable surface with orientation U (unit nvf)



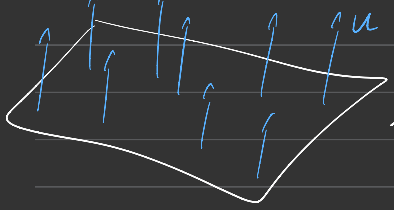


We define the gauss map G as follows

$$G : M \longrightarrow S^2(1) \subset \mathbb{R}^3$$
$$p \longmapsto U(p)$$

Basic Examples

- M = 2-dim plane



$M \xrightarrow{G} S^2$

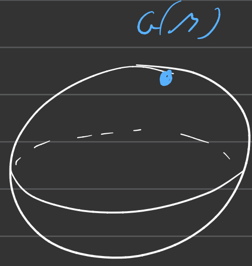
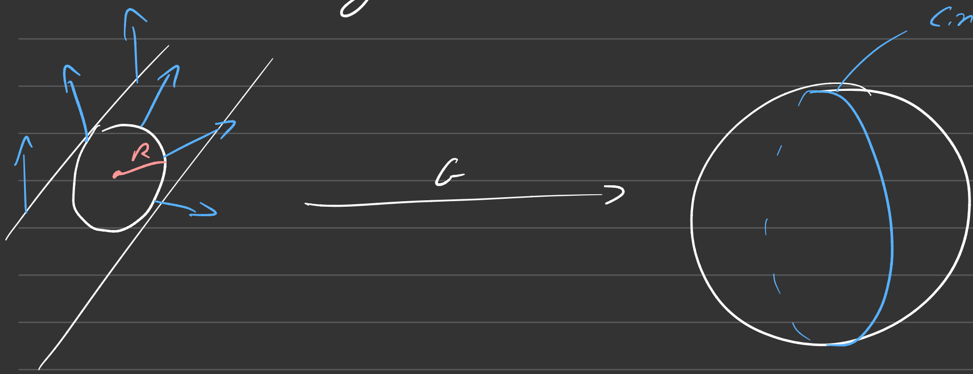
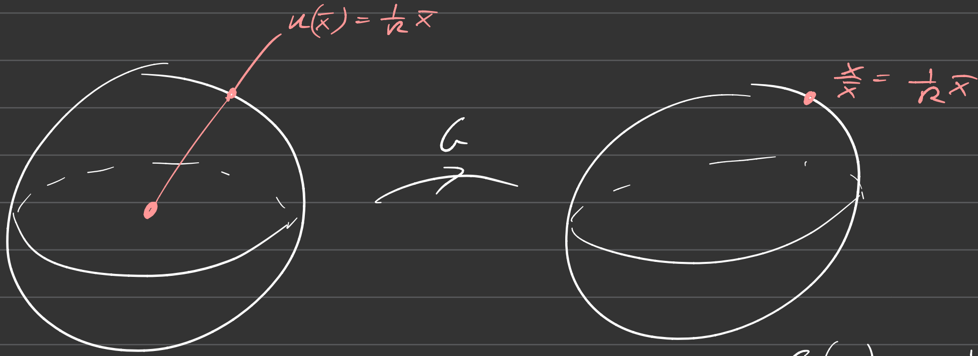


Image of G is a single point

• M = Round Cylinder



• $M = S^2(R)$



$S^2(R)$

$G(\bar{x}) = \frac{1}{R}\bar{x}$

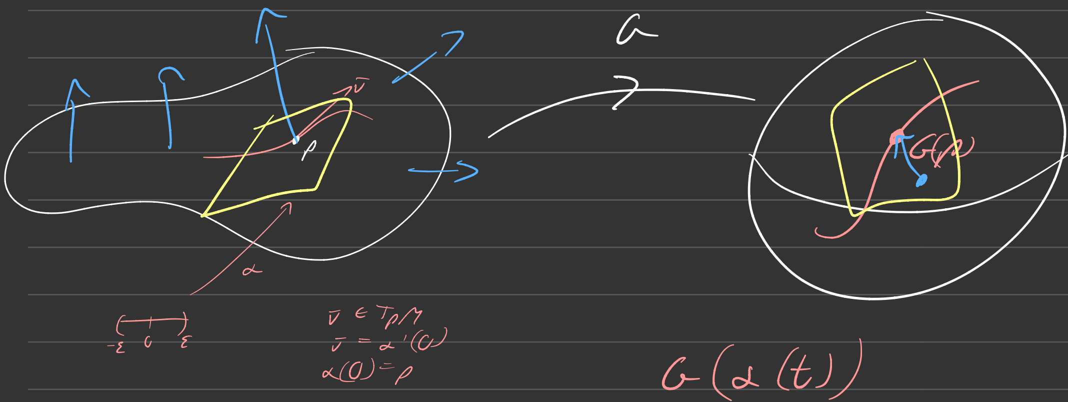
$\mathrm{Im}(G) = S^2(1)$

Differentiating G

u orn on M

$G: M \longrightarrow S^2$

What is DG_p ?



$$\left(DG_p : T_pM \xrightarrow[\text{linear}]{} T_{G(p)}S^2 \right)$$

$$DG_p(\bar v) = \frac{d}{dt}\Big|_{t=0} G(\alpha(t)) \in T_{G(p)}S^2$$

$$= \nabla_{\bar v} U(p)$$

<u>Nice observation</u>

we know that

$$\nabla_{\bar v} U(p) \in T_{G(p)}S^2 = T_pM \quad \text{since } \bar v \perp U(p)$$

$$\parallel$$

$$\{\bar w \in \mathbb{R}^3 : \bar w \perp U(p)\}$$

Thus we obtain a <u>linear</u> <u>operator</u>!

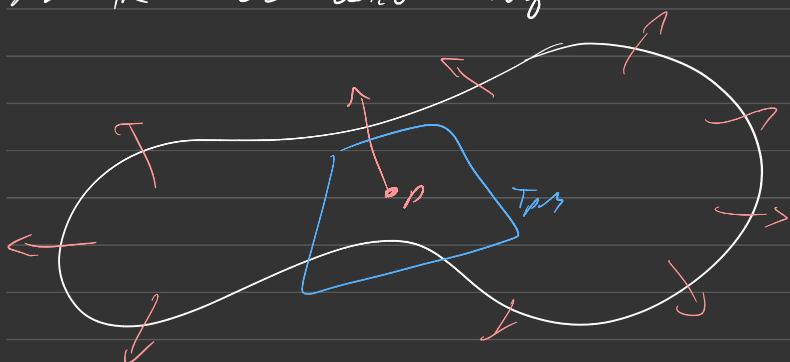
Shape Operator

$$S_p : T_pM \longrightarrow T_pM = T_{G(p)}S^2$$
$$\bar{v} \longmapsto (-DG_p(\bar{v}))$$
$$\qquad\qquad\qquad\qquad = -\nabla_{\bar{v}} U(p)$$

makes certain calculation convenient later

Shape Operator

$M \subset \mathbb{R}^3$ $\quad U$ unit nvf



$$S_p : T_pM \longrightarrow T_pM$$
$$\bar{v} \longrightarrow -\nabla_{\bar{v}} U$$

(equals minus derivative of Gauss map)

Example

$M = S^2(R)$

$U(\bar{p}) = \frac{1}{R}\bar{p}$ (outer pointing vector field)

$\bar{v} \in T_p M$

$$S_p(\bar{v}) = -\nabla_{\bar{v}}\left(\frac{1}{R}\bar{p}\right)$$

$$= -\frac{1}{R}\nabla_{\bar{v}}(p_1, p_2, p_3)$$

$$= -\frac{1}{R}\sum_{i=1}^{3}\bar{v}[p_i]$$

$$= -\frac{1}{R}(v_1, v_2, v_3)$$

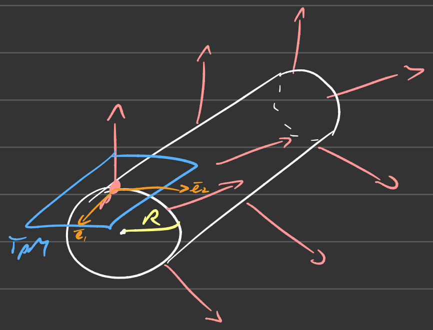
$$= -\frac{1}{R}\bar{v}$$

$$S_p(\bar{v}) = -\frac{1}{R}\bar{v}$$

$$\text{Matrix } S_p = \begin{bmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{bmatrix}$$

Example

M = cylinder of radius R



U outward unit normal vector field

$S_p(\bar{e}_1) = \bar{0}$ since U constant along $\bar{e}_1$ direction

$S_p(\bar{e}_2) = -\frac{1}{2}\bar{e}_2$

$$S_p(\bar{v}) = S_p(v_1\bar{e}_1 + v_2\bar{e}_2)$$

$$= v_1 S_p(\bar{e}_1) + v_2 S_p(\bar{e}_2)$$

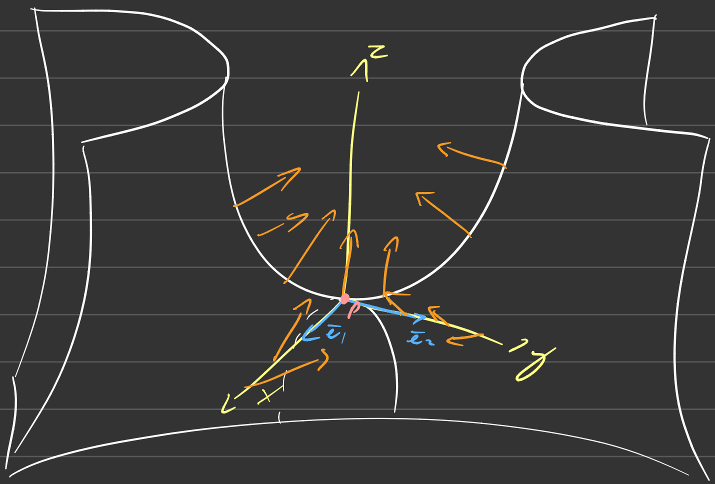
$$= v_1\bar{0} - \frac{1}{R}v_2\bar{e}_2$$

Matrix $S_p = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{bmatrix}$

Example

$M = f^{-1}(0)$ where $f(x,y,z) = z - xy$

$p = \bar{0}$. U spread at $\bar{0}$



$S_p(\bar{e}_1) = \bar{e}_2$

$S_p(\bar{e}_2) = \bar{e}_1$

Matrix $S_p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(If a matrix is symmetric then its eigenvalue will always be real)

Lemma

$p \in M \subset \mathbb{R}^3$, U unit n. on M

$S_p : T_pM \longrightarrow T_pM$ is a symmetric linear operator

This means that $\forall\ \bar{u}, \bar{v} \in T_pM$

$$S_p(\bar{u}) \cdot \bar{v} = \bar{u} \cdot S_p(\bar{v})$$

Proof Later

Consequence

We obtain a symmetric bilinear form

$$\mathrm{II}_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$(\bar{u}, \bar{v}) \longmapsto S_p(\bar{u}) \cdot \bar{v}$$

$$\mathrm{II}_p(\bar{u}, \bar{v}) = \mathrm{II}_p(\bar{v}, \bar{u}) \quad \forall\ \bar{u}, \bar{v} \in T_pM$$

In particular, this means

II_p (and S_p) have all real eigenvalues

II_p is called the second fundamental form to M at p wrt U

Definitions

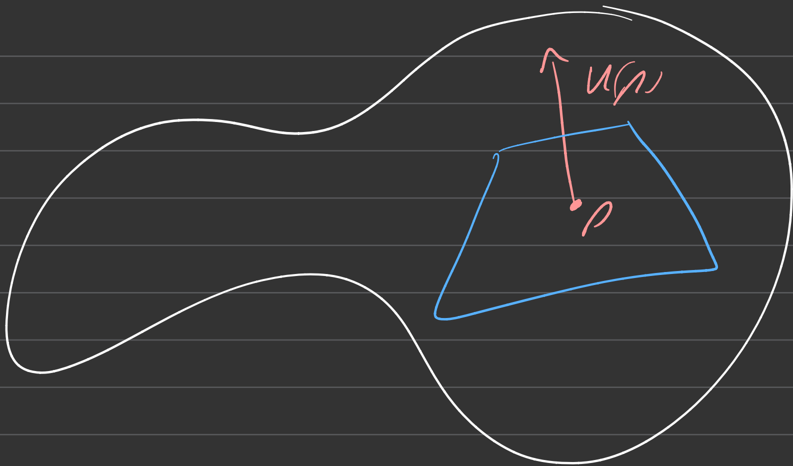
- The eigenvalues of S_p (II_p) are called the principal curvatures of M at p. The corresponding eigendirections are called the principal directions
- The determinant of S_p (II_p) is called the gaussian curvature of M at p
- One half the trace of S_p (II_p) is called the mean curvature of M at p (wrt U)

Notation

Principal Curvatures $\kappa_1(p) \leq \kappa_2(p)$

Gaussian Curvature $K(p) := \det S_p = \kappa_1(p)\,\kappa_2(p)$

Mean Curvature $H(p) := \frac{1}{2} \operatorname{trace} S_p = \frac{1}{2}(\kappa_1(p) + \kappa_2(p))$

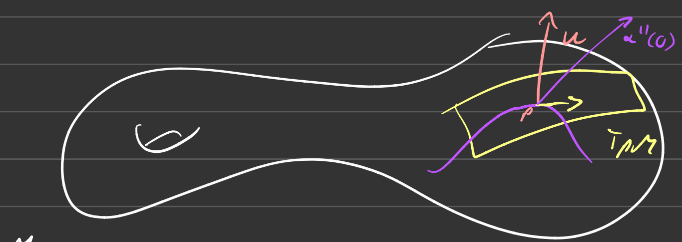


$M \subset \mathbb{R}^3$ surface,

U unvf on M

Shape operator at p

$$S_p : T_pM \longrightarrow T_pM$$
$$\vec{v} \longmapsto \nabla_{\vec{v}} U(p)$$
$$= \frac{d}{dt}\Big|_{t=0} U(\alpha(t))$$



$(-\varepsilon \quad 0 \quad \varepsilon)$

$\alpha(0) = p$

$\alpha'(0) = \vec{v}$

Finally we have the 2nd fundamental form

$$\mathrm{II}_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$(\vec{u},\vec{v}) \longmapsto S_p(\vec{u})\cdot\vec{v}$$

This is a symmetric bilinear form

Suppose $\bar{v} \in T_pM$ is unit length so $|\bar{v}| = 1$

Define $K(\bar{v}) = \mathrm{II}_p(\bar{v},\bar{v})$

This is called the normal curvature at p in the direction $\bar{v}$

Fact

$$K(\vec{v}) = S(\bar{v})\cdot\bar{v}$$

$$= S(\alpha'(0))\cdot\alpha'(0)$$

$$= -\nabla_{\alpha'(0)} U(p) \cdot \alpha'(0)$$

$$= +\alpha''(0)\cdot U(p)$$

$\alpha'(0) \cdot U(p) = 0$

$\alpha'(t) \cdot U(\alpha(t)) = 0$

$\alpha''(0) \cdot U(0) = -\alpha'(0) \cdot U'(\alpha(0))$

$= + \alpha'(0) \cdot S(\alpha'(0))$

<u>NB</u> α'' is perpendicular to $\bar{v}$ but not necessarily parallel to $U(p)$

Let θ denote the smallest angle between $\alpha''(0)$ and $U(p)$

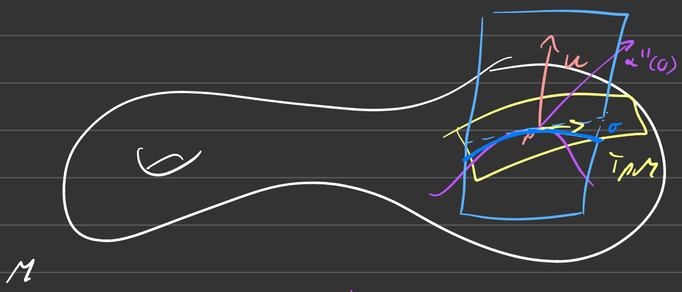
We can say

$K(\bar{v}) = |\alpha''(0)| \, |U(p)| \cos\theta$

$= k_\alpha(0) \cos\theta$

$k_\alpha(0)$ is the curvature of α at 0
Nothing to do with M

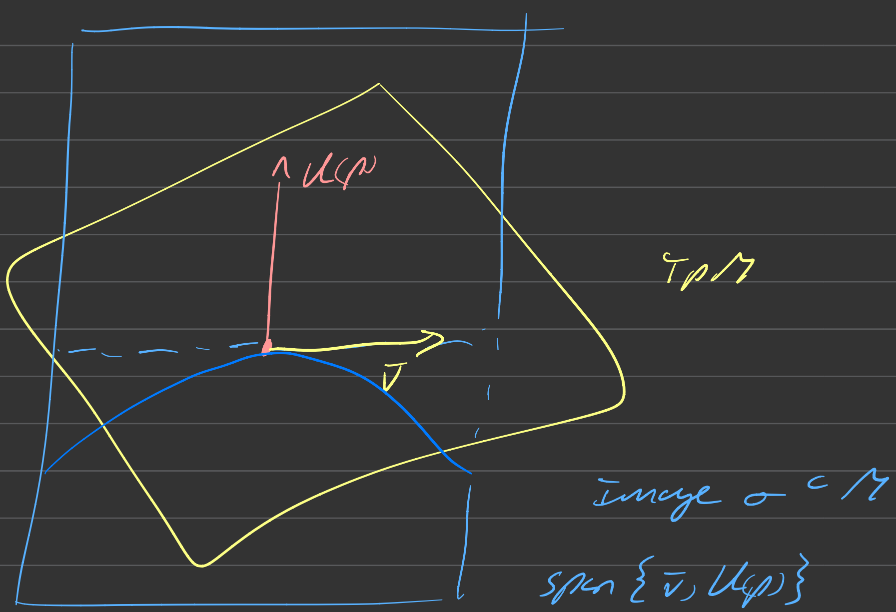
<u>Improvement</u>: Replace α with a "normal slice curve", ie Let σ be the curve running in M obtained by the interface of M with the plane spanned by $U(p)$ and $\bar{v}$



$(-\varepsilon, 0, \varepsilon)$

$\alpha(0) = p$

$\alpha'(0) = \bar{v}$



We can always parametrise

σ so that

$\sigma : (-\varepsilon, \varepsilon) \longrightarrow M$ unit speed

$\sigma(0) = p$

$\sigma'(0) = \bar{v}$

importantly angle $\theta = 0$ or π

Now we have

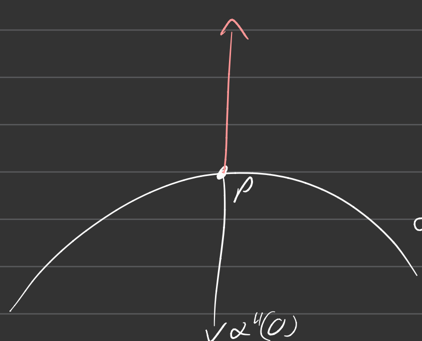
$$K(\bar{v}) = k_\sigma(0) \cos\theta$$

where $\theta = 0$ or π

$$K(\bar{v}) = \pm k_\sigma(0)$$

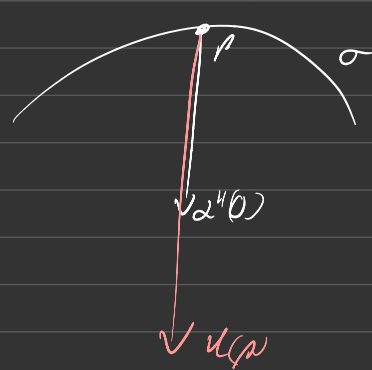
$\theta = \pi$

$K(\bar{v}) = -k_\sigma(0)$



$\theta = 0$

$K(\bar{v}) = k_\alpha(0)$



For each $p \in M$

unit tangent vectors

$K_p : \text{Unit}(T_pM) \longrightarrow \mathbb{R}$

Also easy to see K is continuous

As S^1 is compact, K attains a maximum and a minimum

We call these K_{min} and K_{max}. These are known as ~~the~~ principal curvatures

$$K_{min} \leq K(\bar{v}) \leq K_{max} \quad \forall \ \bar{v} \in \text{Unit}(T_pM)$$

2 cases

Case 1 $\quad K(\bar{v}) = K_{min} = K_{max} = \text{constant} \quad \forall \ \bar{v}$ unit

When this happens the point p is said to be umbilic

Example

$S^2(r)$ round sphere radius r, U outward (inward)

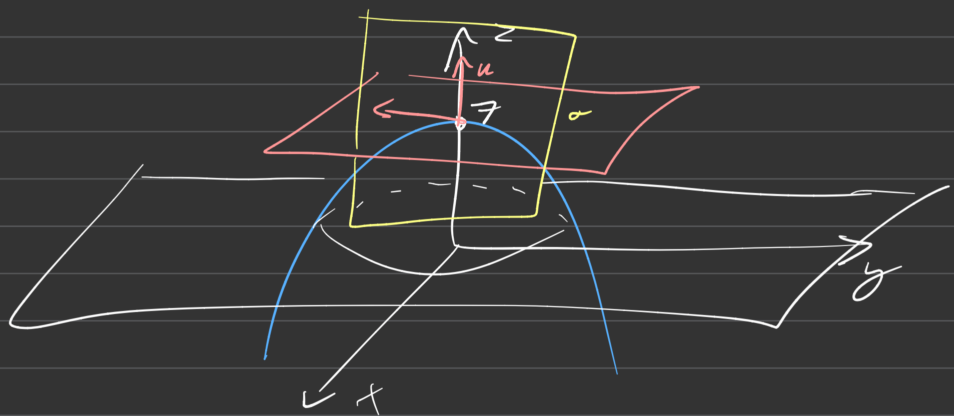
$$p \in S^2; \qquad S_p(\bar{v}) = -\frac{1}{r}\bar{v}$$

$$K(\bar{v}) = S_p(\bar{v}) \cdot \bar{v} = +\frac{1}{r}\bar{v} \cdot \bar{v} = +\frac{1}{r}$$

every point on $S^2(r)$ is umbillic

Example

Paraboloid graph of $f(x,y) = 7 - x^2 - y^2$



$(0,0,7)$ is an isolated umbillic

Case 2

$k_{min} < k_{max}$

In this case we have vectors $\bar{u}_{min}$ and $\bar{u}_{max}$ so that

$$K(\bar{u}_{min}) = k_{min}$$

and $K(\bar{u}_{max}) = k_{max}$

Theorem

The values k_{min} and k_{max} are eigenvalues of S_p and the values spanned by $\bar{u}_{min}$ and $\bar{u}_{max}$ are the respective eigenspaces in T_pM

Since S_p is symmetric the directions $\bar{u}_{min}$, $\bar{u}_{max}$ are orthogonal

---

$p \in M \subset \mathbb{R}^3$, U unit nvf on M

$S_p : T_pM \longrightarrow T_pM$ shape operator at p
$\bar{v} \longmapsto -\nabla_{\bar{v}} U$

$\bar{u}, \bar{v} \in T_pM \qquad II_p(\bar{u}, \bar{v}) = S_p(\bar{u}) \cdot \bar{v} = \bar{u} \cdot S_p(\bar{v})$

$\bar{u} \in Unit(T_pM)$

- Normal curvature at p in direction u $\qquad K(\bar{u}) = II_p(\bar{u}, \bar{v})$
- Gaussian Curvature $\qquad K(p) = \det S_p$
- Mean Curvature $\qquad H(p) = \frac{1}{2} \operatorname{trace} S_p$

As S_p symmetric its eigenvalues K_1, K_2 are real

<u>Case 1</u>

$K_1 = K_2$. In this case

$S_p(\bar{v}) = K_1 \bar{v} = K_2 \bar{v}$

$K(\bar{u}) = K_1$ constant

In this case p is an umbilic point

<u>Case 2</u>

$K_1 < K_2$ (wlog)

<u>Theorem</u>

In this case there are orthogonal unit vectors (eigen-directions) $\bar{u}_1, \bar{u}_2 \in T_pM$ with $S_p(\bar{u}_1) = K_1$, $S_p(\bar{u}_2) = K_2$ and $K_1 \leq K(\bar{u}) \leq K_2 \quad \forall \; \bar{u} \in \text{Unit } T_pM$

<u>Recall</u>

$\bar{u}_1, \bar{u}_2$ are principal directions of M at p w/t U , K_1, K_2 " " curvatures " - - - " "

$V \subseteq \mathbb{R}^n$ dot prod

$L : V \longrightarrow V$ symmetric

$II(\bar{w},\bar{v}) = L(\bar{w})\cdot\bar{v} = \bar{w}\cdot L(\bar{v})$

$\bar{w},\bar{v}$ eigenvectors $L(\bar{w}) = \lambda_1 \bar{w}$, $L(\bar{v}) = \lambda \bar{v}$

$L(\bar{w})\cdot\bar{v} = \bar{w}\cdot L(\bar{v})$

$\Rightarrow \lambda_1 \bar{w}\cdot\bar{v} = \lambda_2 \bar{w}\cdot\bar{v}$

Either $\bar{w}\cdot\bar{v} \neq 0$ and so $\lambda_1 = \lambda_2$

or $\lambda_1 \neq \lambda_2 \Rightarrow \bar{w}\cdot\bar{v} = 0$ //

Proof

$\bar{w} \in$ Unit T_pM

$\bar{w} = \cos\theta\, \bar{w}_1 + \sin\theta\, \bar{w}_2 \qquad \theta \in [0, 2\pi]$

$= c\,\bar{w}_1 + s\,\bar{w}_2$

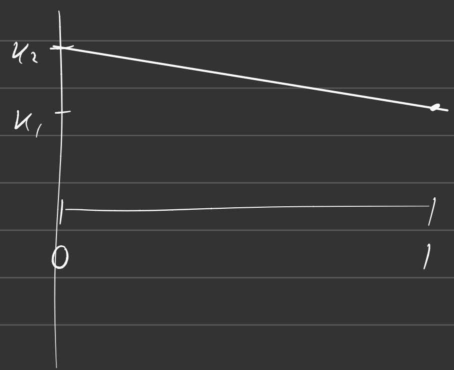
$K(\bar{w}) = II_p(\bar{w},\bar{w})$

$$= II_p(c\bar{u}_1 + s\bar{u}_2,\ c\bar{u}_1 + s\bar{u}_2)$$

$$= c^2 \kappa(\bar{u}_1) + s^2 \kappa(\bar{u}_2) \qquad (\bar{u}_1 \cdot \bar{u}_2 = 0)$$

$$= c^2 \kappa_1 + s^2 \kappa_2$$

$$= t\kappa_1 + (1-t)\kappa_2 \qquad (t = c^2)$$



$0 < \theta < \frac{\pi}{2}$

Thus $\kappa_{max} = \kappa_2$

$\kappa_{min} = \kappa_1$

Observation

Can extend normal curvature to all of $T_pM \setminus \{\bar{0}\}$

by $\quad K(\bar{v}) = \dfrac{\mathrm{II}_p(\bar{v},\bar{v})}{\bar{v}\cdot\bar{v}}$

Note

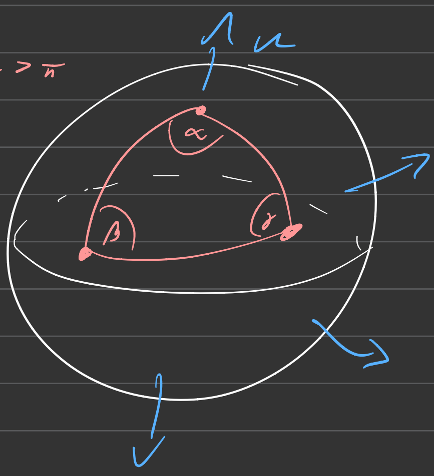
$K(p) = k_1(p)\,k_2(p) = \det S_p$

$H(p) = \frac{1}{2}\left(k_1(p) + k_2(p)\right) = \frac{1}{2}\operatorname{trace} S_p$

Examples

- $S^2(r)$, U = outward unit normal

$\alpha+\beta+\gamma > \pi$



$S_p(\bar{v}) = -\frac{1}{r}\bar{v}$

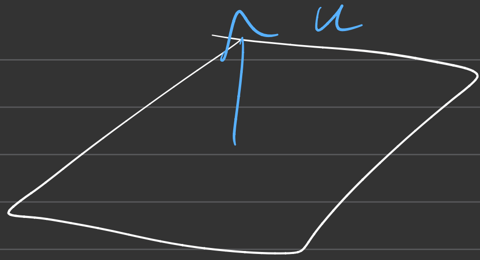
$k_1 = k_2 = -\frac{1}{r}$

$K(p) = \frac{1}{r^2} > 0$

$H(p) = -\frac{1}{r} = \frac{1}{2}\left(-\frac{1}{r} - \frac{1}{r}\right)$

Area $\Delta = \alpha+\beta+\gamma-\pi$

- Plane

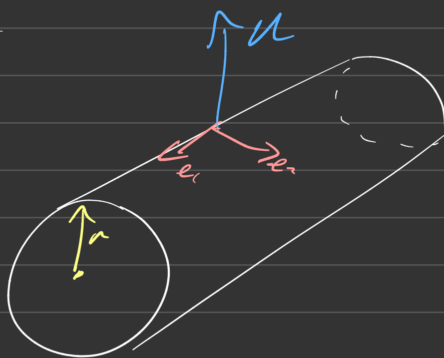


$S_p(\bar{v}) = \bar{0}$

$K(p) = 0$

$H(p) = 0$

- Round Cylinder



$\bar{v} = v_1 \bar{e}_1 + v \bar{e}_2$

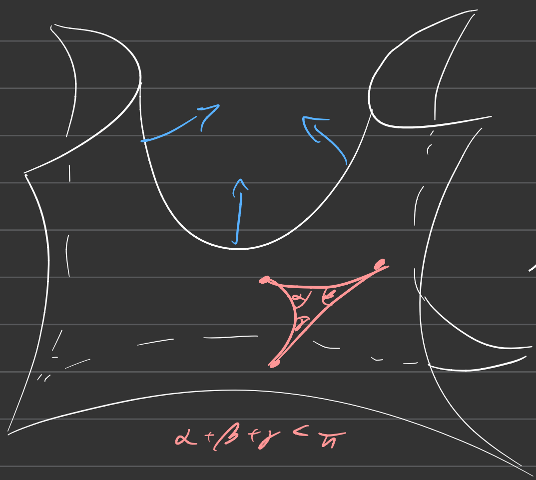
$S_p(\bar{v}) = 0 v_1 e_1 - \frac{1}{r} v_2 e_2$

$K_{min}(p) = -\frac{1}{r}$ $\quad K_{max}(p) = 0$

$K_p = -\frac{1}{r}(0) = 0$

$H(p) = -\frac{1}{2r}$

• Graph of $f(x,y) = xy$ $\quad p = \bar{0}$



$S_p(\bar{e}_1) = \bar{e}_2$

$S_p(\bar{e}_2) = \bar{e}_1$

Matrix $S_p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\kappa^2 - 1 = 0$

$\Rightarrow \kappa = \pm 1$

$\kappa_1 = -1$, $\kappa_2 = 1$ principal curvatures

$K(0) = -1 < 0$

$H(0) = 0$

• There exists a surface with $K \equiv -1$, (hyperbolic 2-space)

$\pi - (\alpha + \beta + \gamma) = \text{Area } \Delta$

Recap

$M \subset \mathbb{R}^3$ $\quad U$ surf

$S_p : T_pM \longrightarrow T_pM$ $\quad$ Shape operator

$\bar{v} \longmapsto -\nabla_{\bar{v}}(U)$

$\mathrm{II}_p : T_pM \times T_pM \longrightarrow \mathbb{R}$

$(\bar{u}, \bar{v}) \longmapsto S_p(\bar{u}) \cdot \bar{v} = \bar{u} \cdot S_p(\bar{v})$

$\bar{u}$ unit length in T_pM

$K(\bar{u}) = \mathrm{II}_p(\bar{u}, \bar{u})$ $\quad$ normal curvature

Theorem

Eigenvalues of S_p (if distinct) are min and max values of K

$$K_{min} \leq K(\bar{u}) \leq K_{max}$$

In case $K_{min} = K_{max}$ we say p is an umbilic point of M

Corresponding eigen-directions (in case $K_{min} \leq K_{max}$) are called principal directions

Gaussian Curvature

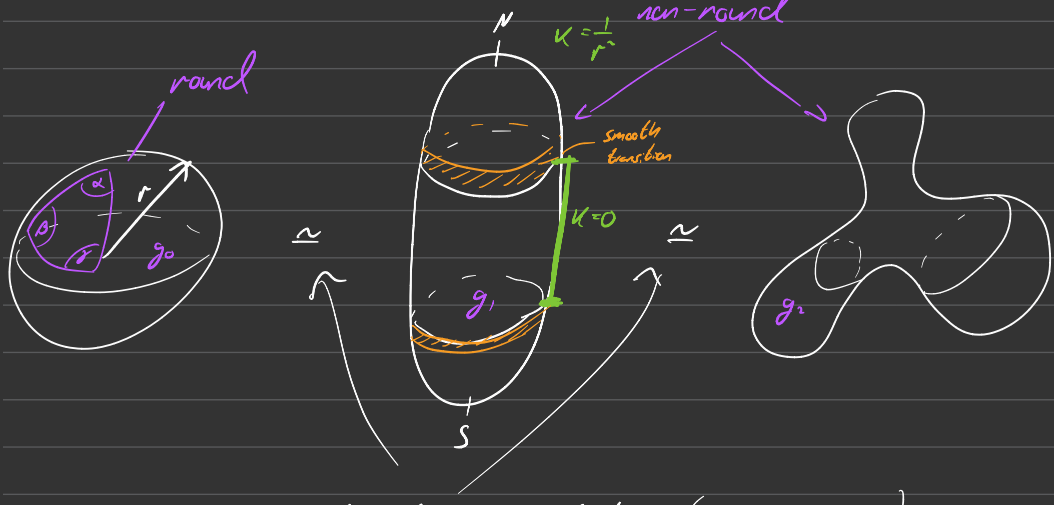
$$K(p) = \det S_p = k_{min}(p)\, k_{max}(p)$$

Mean Curvature

$$H(p) = \frac{1}{2} \operatorname{trace} S_p = \frac{1}{2}\left(k_{min}(p) + k_{max}(p)\right)$$

Gaussian Curvature

Topological type:



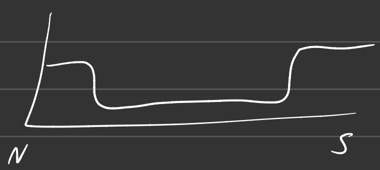
Topologically equivalent (geometrically distinct)

Can we use a surface a constant curvature metric

g_0 = round metric of radius r

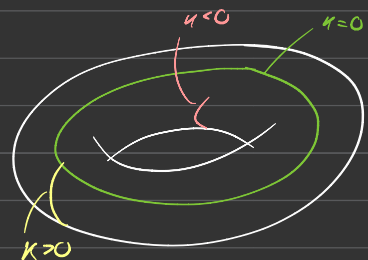
$K(g_0) = \frac{1}{r^2}$

$K(g_1) = ?$



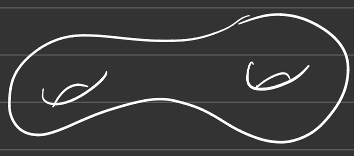
S^2 admits constant $K \equiv c > 0$ metric

In $\mathbb{R}^3$, no obvious constant curvature metric on T^2



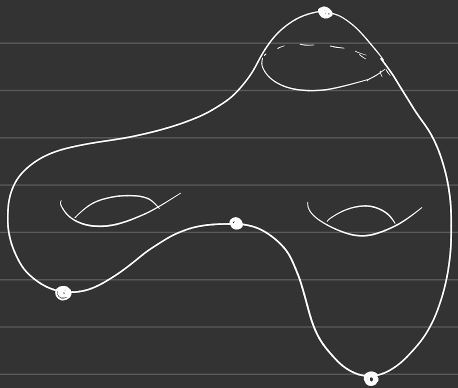
Problem even worse for

$T^2 \# T^2$ etc



Theorem

A compact smooth surface in $\mathbb{R}^3$ must have a point with positive gaussian curvature.

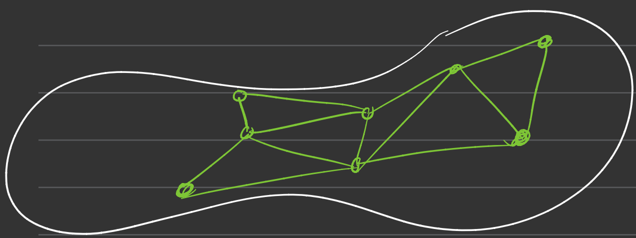


Gauss-Bonnet Theorem

M surface (in $\mathbb{R}^3$ - though not actually necessary)

$K : M \longrightarrow \mathbb{R}$

Suppose we have a finite triangulation (polygonilisation) of M



v = num of vertices

e = num of edges

f = num of faces

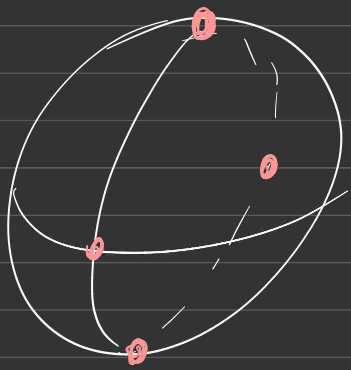
$\chi(M)$ = Euler number of M (independent of triangulation)

$= v - e + f$

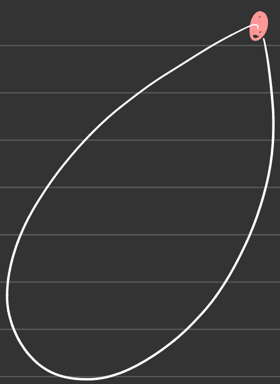
$$\boxed{\frac{1}{2\pi}\int_M K\,dA = \chi(M)}$$

$K > 0 \Rightarrow \chi(M) > 0$

Example

- 

$v = 4$
$e = 6$
$f = 4$

- 

$v = 1$
$e = 0$
$f = 1$

Example

- $\chi(S^2) = 2$
- $\chi(T^2) = 0$

$\chi(T^2 \# T^2) = -2$

$\chi(\underbrace{T^2 \# \dots \# T^2}_{n\text{-times}}) = 2 - 2n$

Recap

$M \subset \mathbb{R}^3$ $\quad U$ $\quad$ surf

$$S_p : T_pM \longrightarrow T_pM \qquad \text{Shape operator}$$
$$\bar{v} \longmapsto -\nabla_{\bar{v}}(U)$$

$$\mathbb{II}_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$(\bar{u}, \bar{v}) \longmapsto S_p(\bar{u}) \cdot \bar{v} = \bar{u} \cdot S_p(\bar{v})$$

$\bar{u}$ unit length in T_pM

$$K(p) = \det S_p$$

$$H(p) = \frac{1}{2} \operatorname{trace}(S_p)$$

3 useful formulae

Lagrange: $\bar{u}, \bar{v} \quad \bar{a}, \bar{b} \in \mathbb{R}^3$

$$(\bar{u} \times \bar{v}) \cdot (\bar{a} \times \bar{b}) = \det \begin{pmatrix} \bar{u} \cdot \bar{a} & \bar{u} \cdot \bar{b} \\ \bar{v} \cdot \bar{a} & \bar{v} \cdot \bar{b} \end{pmatrix}$$

(1) $\bar{u}, \bar{v} \in T_pM$

Assume $\bar{u}, \bar{v}$ linearly independent

$$S(\bar{u}) \times S(\bar{v}) = K(p)(\bar{u} \times \bar{v}) \qquad (S_p = S)$$

(2) $$(S(\bar{u}) \times \bar{v}) + (\bar{u} \times S(\bar{v})) = 2H(p)(\bar{u} \times \bar{v})$$

Exercise: Prove these formulae

These can be rewritten as

(1) $$K(p) = \frac{(S(\bar{u}) \times S(\bar{v})) \cdot (\bar{u} \times \bar{v})}{|\bar{u} \times \bar{v}|^2}$$

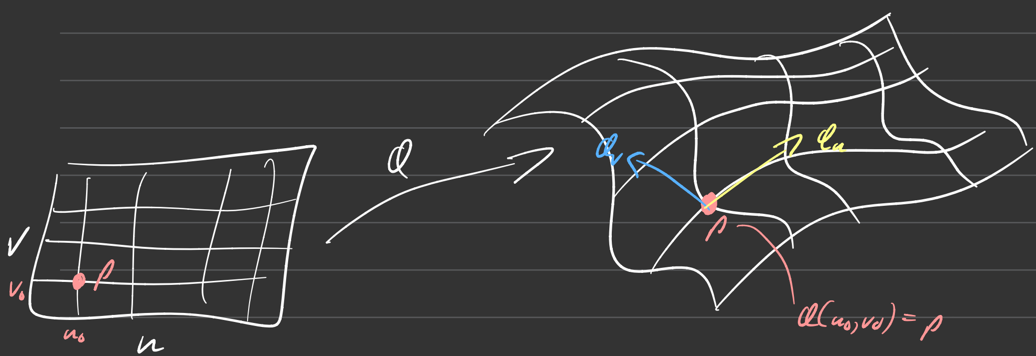
$$= \frac{\det \begin{bmatrix} \mathrm{II}(\bar{u}, \bar{u}) & \mathrm{II}(\bar{u}, \bar{v}) \\ \mathrm{II}(\bar{v}, \bar{u}) & \mathrm{II}(\bar{v}, \bar{v}) \end{bmatrix}}{(\bar{u} \cdot \bar{u})(\bar{v} \cdot \bar{v}) - (\bar{u} \cdot \bar{v})^2}$$

$H(p)$

$$= \frac{\det\begin{bmatrix} \mathrm{II}(\bar{u},\bar{u}) & \mathrm{II}(\bar{u},\bar{v}) \\ \bar{u}\cdot\bar{v} & \bar{v}\cdot\bar{v}\end{bmatrix} - \det\begin{bmatrix} \bar{u}\cdot\bar{u} & \bar{u}\cdot\bar{v} \\ \mathrm{II}(\bar{u},\bar{v}) & \mathrm{II}(\bar{v},\bar{v})\end{bmatrix}}{2(\bar{u}\cdot\bar{u})(\bar{v}\cdot\bar{v}) - 2(\bar{u}\cdot\bar{v})^2}$$

Suppose $\alpha : \mathcal{D} \longrightarrow \mathbb{R}^3$ is a coord patch

[$\mathcal{D}$ is a rectangle with co-ords u, v]



$E = \alpha_u \cdot \alpha_u\ ,\quad F = \alpha_u \cdot \alpha_v\ ,\quad G = \alpha_v \cdot \alpha_v$

$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$

"I" $=$

$$\bar{w} = w_1 \mathbf{x}_u + w_2 \mathbf{x}_v$$

$$I(\bar{w}, \bar{z}) = [w_1, w_2] \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

<u>Useful excercise</u>

$$\mathcal{U} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$$

$$S(\bar{w}) = v_1 S(\mathbf{x}_u) + v_2 S(\mathbf{x}_v)$$

$$S(\bar{w}) = -\nabla_{\bar{w}} \mathcal{U} = ?$$

Recall, when $\bar{w} = \alpha'(0)$ for $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ $\alpha(0) = p$

$$\mathrm{II}(\bar{w}, \bar{w}) = \mathrm{II}(\alpha', \alpha') = \alpha'' \mathcal{U}$$

<u>Exercise</u>

$$\mathrm{II}(\mathbf{x}_u, \mathbf{x}_u) = \mathbf{x}_{uu} \cdot \mathcal{U}, \quad \mathrm{II}(\mathbf{x}_u, \mathbf{x}_v) = \mathbf{x}_{uv} \cdot \mathcal{U}$$

$$\mathrm{II}(\mathbf{x}_v, \mathbf{x}_v) = \mathbf{x}_{vv} \cdot \mathcal{U} \qquad \mathrm{II}(\mathbf{x}_v, \mathbf{x}_u) = \overset{\parallel}{\mathbf{x}_{vu}} \cdot \mathcal{U}$$

<u>Tradition</u>

$$l = \mathrm{II}(\mathbf{x}_u, \mathbf{x}_u) = \mathbf{x}_{uu} \cdot \mathcal{U}$$

$$m = \mathrm{II}(\mathbf{x}_u, \mathbf{x}_v) = \mathbf{x}_{uv} \cdot \mathcal{U}$$

$$n = \mathrm{II}(\mathbf{x}_v, \mathbf{x}_v) = \mathbf{x}_{vv} \cdot \mathcal{U}$$

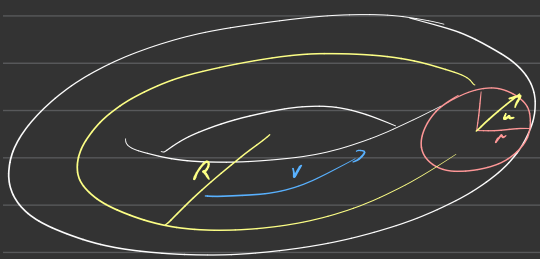
Now our earlier formulae (1) (2) become

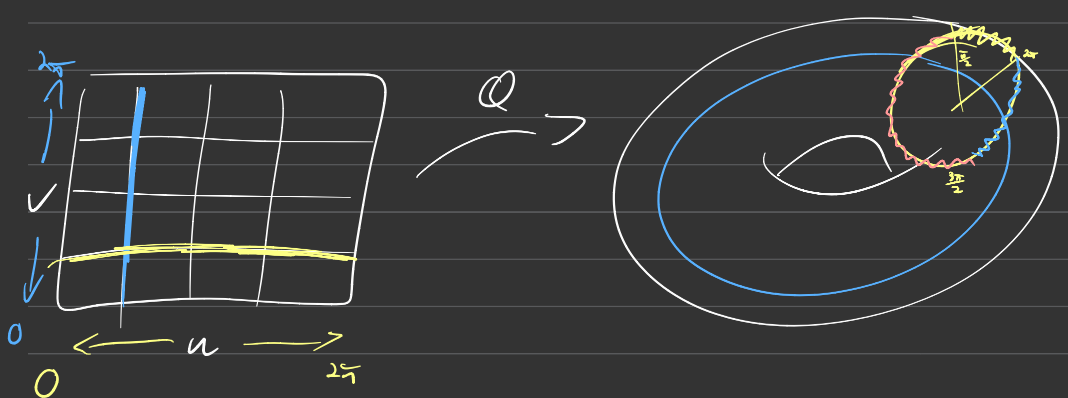
$$K(p) = \frac{ln - m^2}{EG - F^2}$$

$$H(p) = \frac{Gl + En - 2Fm}{2(EG - F^2)}$$

Example

Torus $\qquad \Omega = \underset{(u\ \ ,}{(0, 2\pi)} \times \underset{v\ )}{(0, 2\pi)}$





$$\mathbf{x}(u,v) = \begin{bmatrix} (R+r\cos u)\cos v \\ (R+r\cos u)\sin v \\ r\sin v \end{bmatrix}$$

Calculate yourselves U, $\mathbf{x}_u$, $\mathbf{x}_v$, $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, $\mathbf{x}_{vv}$

Metric

$$ds^2 = r^2 du^2 + (R+r\cos u)^2 dv^2$$

(E, F, G) ✓

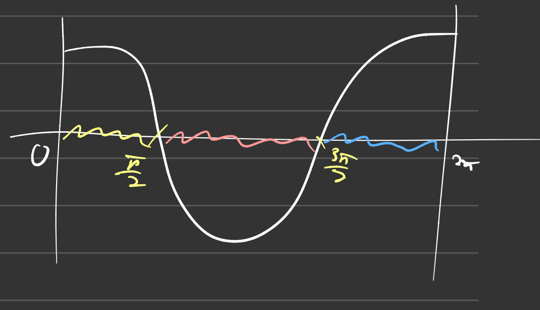
Calculat $\quad l = \mathbf{x}_{uu} \cdot U$

$$m = \mathbf{x}_{uv} \cdot U$$

$$n = \mathbf{x}_{vv} \cdot U$$

Finally

$$K = \frac{\cos u}{r(R + r\cos u)}$$



Computational Techniques

Recap

$\mathbf{x} : D \longrightarrow \mathbb{R}^3$ a coord patch
(u,v)

$E = \mathbf{x}_u \cdot \mathbf{x}_u \quad , \quad F = \mathbf{x}_u \cdot \mathbf{x}_v \quad , \quad G = \mathbf{x}_v \cdot \mathbf{x}_v$

$$U = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$$

$\ell = \mathrm{II}(\mathbf{x}_u, \mathbf{x}_u) = \mathbf{x}_{uu} \cdot U$

$m = \mathrm{II}(\mathbf{x}_u, \mathbf{x}_v) = \mathbf{x}_{uv} \cdot U$

$n = \mathrm{II}(\mathbf{x}_v, \mathbf{x}_v) = \mathbf{x}_{vv} \cdot U$

$$K = \frac{\ell n - m^2}{EG - F^2} \quad , \quad H = \frac{G\ell + En - 2Fm}{2(EG - F^2)}$$

Example

Monge Patch

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ smooth function

$\alpha(u,v) = (u, v, f(u,v))$

$\alpha_u (1, 0, f_u)$, $\alpha_v = (0, 1, f_v)$

$$U = \frac{\alpha_u \times \alpha_v}{|\alpha_u \times \alpha_v|} = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}$$

$\alpha_{uu} = (0, 0, f_{uu})$

$\alpha_{uv} = (0, 0, f_{uv})$

$\alpha_{vv} = (0, 0, f_{vv})$

$E = 1 + f_u^2$ $G = 1 + f_v^2$

$F = f_u f_v$

$$l = \alpha_{uu} \cdot U = \frac{f_{uu}}{\sqrt{1 + f_u^2 + f_v^2}}$$

$$m = \frac{f_{uv}}{\sqrt{1+f_u^2+f_v^2}}$$

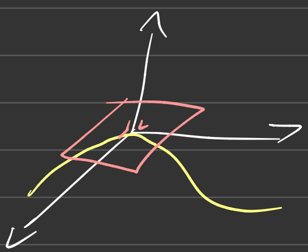
$$n = \frac{f_{vv}}{\sqrt{1+f_u^2+f_v^2}}$$

$$K = \frac{1}{(1+f_u^2+f_v^2)} \frac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2)(1+f_v^2) - f_u^2 f_v^2}$$

$$= \frac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2} = \det(d^2 f)$$

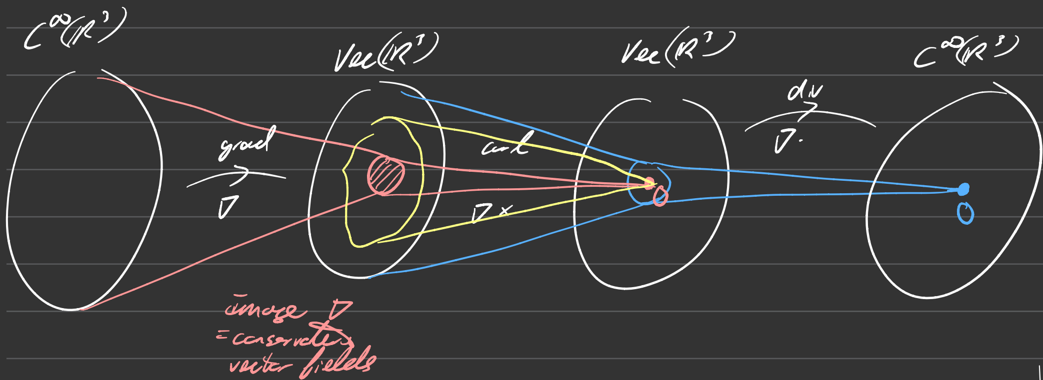
$$f_u(0,0) = f_v(0,0) = 0$$

$$f(0,0) = 0$$



$C^\infty(\mathbb{R}^3)$ = smooth functions $\mathbb{R}^3 \to \mathbb{R}$

$Vec(\mathbb{R}^3)$ = vec fields



Last Time

Patch computations

$$\mathbf{x} : D \longrightarrow \mathbb{R}^3$$

$$(u,v) \longmapsto (x_1(u,v), x_2(u,v), x_3(u,v))$$

$$E = \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v$$

$$U = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$$

$$l = \mathbf{x}_{uu} \cdot U, \quad m = \mathbf{x}_{uv} \cdot U, \quad n = \mathbf{x}_{vv} \cdot U$$

$$K = \frac{ln - m^2}{EG - F^2}, \quad H = \frac{Gl + En - 2Fm}{2(EG - F^2)}$$

Monge Patch

$$\mathbf{x}(u,v) = (u, v, f(u,v))$$

$$K = \frac{f_{uu} f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}$$

$$\nabla f = \bar{0}$$

$K > 0$  loc max / min

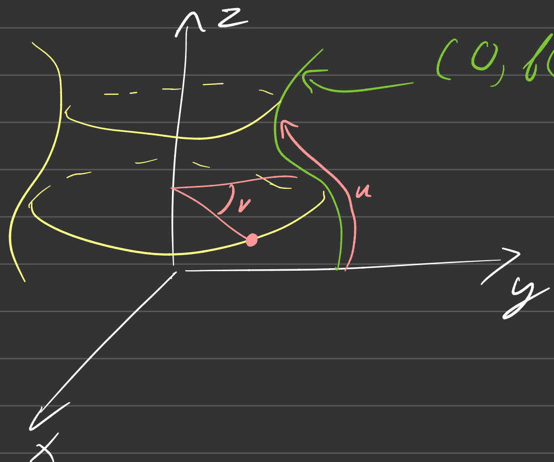
$K < 0$  neither max / min

$K = 0$ 

$H = ?$

Example

Surface of Revolution



$(0, f(s), g(s))$ - curve

Assume unit speed

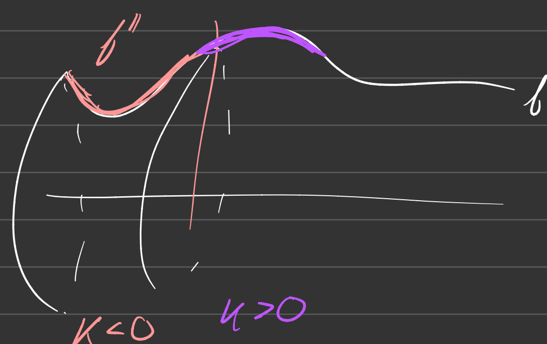
$f'(s)^2 + g'(s)^2 = 1$

Patch $\mathbf{x}(u,v) = (\cos v \cdot f(u), \sin v \cdot f(u), g(u))$

Exercise

Compute K, H

$$K = \frac{-f''(u)}{f(u)}$$



Example

Surfaces as level sets

$f : \mathbb{R}^3 \to \mathbb{R}$ smooth

$f^{-1}(c)$ regular level set

This is a surface

∇f is a non vanishing normal vector field

$U := \frac{\nabla f}{|\nabla f|}$ unit nvf

More generally suppose $M \subset \mathbb{R}^3$ is a surface and Z is a non vanishing normal vector field

Define $U = \frac{Z}{|Z|}$

S = shape operator

V any vector field tangent on M

$$S(V) = -\nabla_V U = -\nabla_V \frac{Z}{|Z|}$$

$$= -\frac{1}{|z|} \nabla_V (z) - \overbrace{\underbrace{\left(V\left[\frac{1}{|z|}\right]\right)}_{\text{scalar}} z}^{\text{normal}}$$

$\forall$ V, W lin indep tangent vector fields to M

($V \times W$ non vanishing)

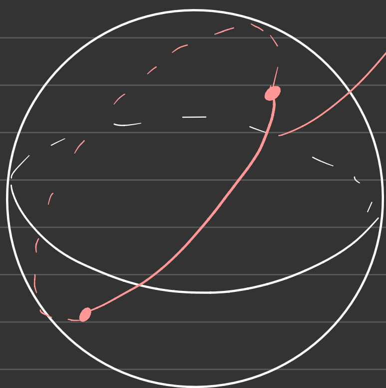
$$K = \frac{z \cdot \nabla_V z \times \nabla_W z}{|z|^4}$$

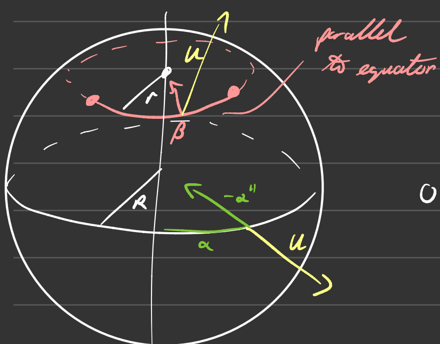
$$H = -z \cdot \left(\frac{(\nabla_V z \times W) + (V \times \nabla_W z)}{2|z|^3} \right)$$

Geodesics

$M \subset \mathbb{R}^3$

U unit normal vector field on M





$0 < r < R$

α'' parallel to U $\qquad \alpha'' = \kappa N_\alpha$

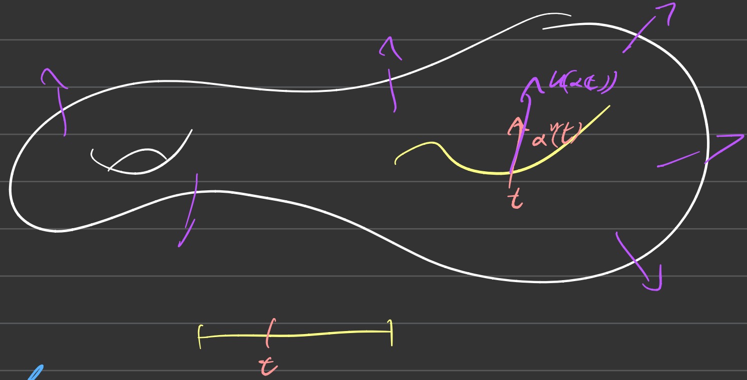
β'' not parallel to U

Geodesics

$M \subset \mathbb{R}^3$ surface

U unit nvf on M

$\alpha : I \to M$, a smooth is a geodesic if $\forall\, t \in I$, $\alpha''(t)$ is parallel to $U(\alpha(t))$



Example

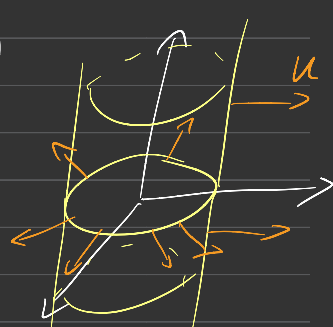
$M =$ cylinder

$= \{ (x,y,z) \mid x^2 + y^2 = 16,\ z \in \mathbb{R} \}$

$\alpha(t) = (4\cos(2t+1), 4\sin(2t+1), 3t+4)$

$\alpha'(t) = (-8\sin(2t+1), 8\cos(2t+1), 3)$

$\alpha''(t) = (-16\cos(2t+1), -16\sin(2t+1), 0)$



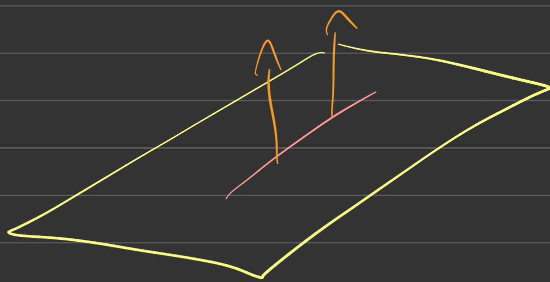
$\alpha''(t)$ parallel to unit normal $U = (\cos t, \sin t, 0)$

~~Example~~

M = plane in $\mathbb{R}^3$

$\alpha(t) = (p_1 + q_1 t,\ p_2 + q_2 t,\ p_3 + q_3 t)$

$\alpha''(t) = (0, 0, 0)$



Exercise

Show that any geodesic $\alpha : I \to M \subset \mathbb{R}^3$ has constant speed

Hint: show $\alpha'(t) \cdot \alpha'(t) = \text{const}$

Proof $\alpha'(t) \cdot \alpha'(t) = c(t)$

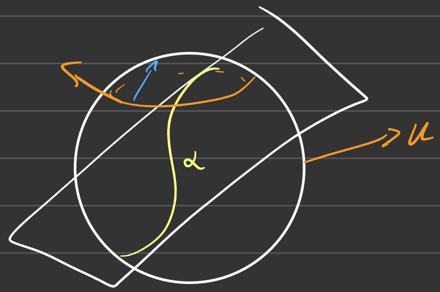
$2\alpha'(t) \cdot \alpha''(t) = c'(t)$

$= 2\alpha'(t) \cdot g(t) U(t) \equiv 0$

Exercise

Characterise geodesics of the $S^2(R)$, $R > 0$

Answer



Geodesics on $S^2(R)$ form parts of great circles

Shape operator $S_p(\bar{v}) = \pm \frac{1}{R} \bar{v} = \pm \nabla_{\bar{v}} U$

$$G : S^2 \longrightarrow S^2$$
$$\bar{x} \longmapsto \frac{1}{R} \bar{x} = U(x)$$

$$S_p(\alpha'(t)) = \pm \frac{1}{R} \alpha'(t) = \pm \nabla_{\alpha'(t)} U = \frac{d}{dt} U(\alpha(t))$$

$$\alpha''(t) = \kappa(t) N(t) \parallel U(t)$$

$$T' =$$

$$N' = -\kappa T \qquad + \tau B$$

$$B' =$$

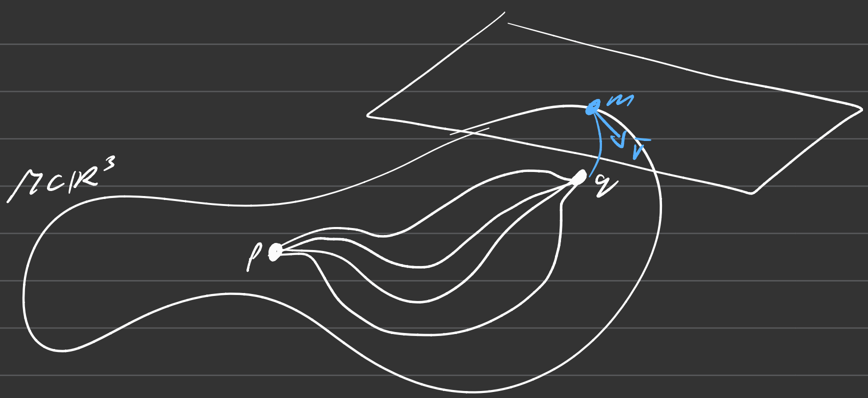
$$N'(t) = \pm \nabla_{\alpha'(t)} U(t)$$

$$\underset{-\kappa T + \tau B}{\parallel} \qquad \pm \frac{1}{R}\alpha'(t) = \pm \frac{1}{R} T(t)$$

$\Rightarrow \tau = 0$

$\Rightarrow \kappa = \frac{1}{R}$ constant

$\Rightarrow$ This curve is a circle



$$d_M(p,q) = \inf_{\gamma} \{ \text{lengths } \gamma : [0,1] \to M : \gamma(0) = p, \gamma(1) = q \}$$

Provided M complete, infimum is attained by some γ and this is part of a geodesic

$M = S^2$